Unconventional Monetary Policy and Funding Liquidity Risk*

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Abstract

This article investigates the efficiency of different monetary policies to stabilize asset prices in a liquidity crisis. We propose a macro-finance model featuring heterogeneous banks subject to funding liquidity risk. When banks are well capitalized, they have access to money markets and efficiently mitigate funding shocks. When bank capital is low, an endogenous haircut spiral between declining asset prices and funding risks arises. The central bank can partially counter these dynamics with monetary policies. Liquidity injection and discount window policies help alleviate stresses in the traditional banking sector but fail to reach to the shadow banking sector. Large-scale asset purchase policy (LSAP) decreases the stock of funding risks through a general equilibrium effect and therefore has a larger reach in the economy. If the shadow banking sector is large, LSAP may be necessary to stabilize asset prices.

Keywords: Monetary Policy, Liquidity Risk, Quantitative Easing, Money Market, Excess Reserves, Shadow Banks.

JEL Classifications: E43, E44, E52, G12

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1 Introduction

In reaction to the financial crisis of 2008, central banks drastically extended their policy toolbox. For example, the Federal Reserve quadrupled the size of its balance sheet by including large amounts of mortgage-backed securities, long-term Treasuries, various liquidity facilities, and swap loans to foreign central banks. The difficulty for central banks to alleviate high degrees of funding risks—the risk of not being able to raise new funds to repay maturing debt—in the non-bank part of the financial sector is often referred as a key reason behind the use of unconventional monetary policies (Bernanke, 2009). Ten years after the crisis, central banks are discussing how to unwind these policies and move to the “new normal” with the following questions in mind: how and when should these new tools be used in the future?

In most of the macroeconomic literature, these policies are undifferentiated and these questions cannot be tackled. To fill this gap, we build an intermediary macro-finance model in the vein of He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014) with two additional features and three explicitly distinct monetary policies. Our first addition is to assume that financial intermediaries are subject to funding shocks and have to solve a liquidity management problem in the spirit of Bianchi and Bigio (2014) and Schneider and Piazzesi (2015). The effects of these funding shocks vary as the economy can enter into a liquidity crisis regime in which money markets—where short-term financial instrument are traded to mitigate liquidity shocks—are impaired and asset prices drop. Our second addition is to introduce shadow banks that only differ from traditional banks by not having access to public sources of liquidity.

The model provides a tractable environment in which the central bank can counteract adverse dynamics by reducing funding liquidity risks in three different ways. First, by increasing the supply of excess reserves to banks (liquidity injection policy), the central bank creates an ex-ante buffer in banks’ balance sheet to absorb funding shocks. Second, by providing access to emergency liquidity facilities (lender of last resort policy), the central bank provides an ex-post relief of the impact of funding shocks. Third, by buying and

\footnote{This assumption is in line with the definition of shadow banks of Adrian and Ashcraft (2012): “While shadow banks conduct credit and maturity transformation similar to traditional banks, shadow banks do so without the direct and explicit public sources of liquidity and tail risk insurance via the Federal Reserve discount window and the Federal Deposit Insurance Corporation (FDIC) insurance.”}
holding risky long-term securities (asset purchase policy\(^2\)), the central bank removes funding risk from the market. For these three policies, the critical assumption that empowers the central bank is its ability to create reserves that is the ultimate means of settlement in the economy.

The first contribution of this article is to provide a tractable model linking funding risks—on the liability side of the balance sheet of financial institutions—to asset prices through the balance sheet of financial intermediaries. In our model, intermediaries engage in liquidity transformation by holding assets that are less liquidity than their liabilities. After a realization of a negative funding shock, an intermediary has to cover a funding gap—the difference between illiquid assets and after-shock funding—by either acquiring funding in money markets (at a negligible cost) or to sell securities at a fire-sale price (at a high cost). Due to information asymmetry, money market lenders require their counterparty to post a sufficient amount of securities as collateral to secure the trade. This assumption endogenously creates two regimes in the economy. In normal times, banks can use money markets efficiently to avoid a costly fire-sale of assets. Funding liquidity risk is therefore low and does not show up in the aggregate pricing kernel. In a crisis, volatility may force margins to become so high that overall available collateral falls short of the requirements to access money markets (a mechanism akin to the haircut spiral in Brunnermeier and Pedersen, 2009). Because financial intermediaries take into account their funding structure when pricing securities, an increase in this funding liquidity risk affect asset prices negatively.

We use the model to investigate the efficiency of different monetary policies in various liquidity regimes (with and without well-functioning money markets) and under different financial structures (size of the shadow banking sector). As in the monetary policy implementation literature (Frost, 1971; Poole, 1968), we assume that central bank reserves are used for interbank settlement. By holding reserves, banks can reduce their exposure to funding risk. We show how this non-pecuniary benefit of holding reserves break Wallace’s (1981) neutrality such that monetary policies affect asset prices and macro variables by reducing the aggregate level of funding liquidity risk. This result applies to liquidity injections, lender of last resort policy and asset purchase policy. Both injecting reserves and

\(^2\)We use the term asset purchase policy rather than the more common Quantitative Easing as the latter is used ambiguously to refer to both buying long term assets (on the asset side of the central bank’s balance sheet) or the corresponding extension the supply of reserves (on the liability side).
lowering the cost of the discount window helps directly alleviating the liquidity risk in the traditional banking sector but fails to reach to the shadow banking sector. In contrast, as the central bank buys and holds illiquid assets, it destroys stocks of funding risks from the economy as a consequence of the central bank not facing liquidity risk due to its ability to issue reserves. This latter form of policy has the advantage of operating through a general equilibrium channel with a broader reach.

Our analysis concludes that, in the presence of a sizeable shadow banking sector and impaired money markets, liquidity injection and lender of last resort policies may not be sufficient to alleviate funding stresses. Stabilizing asset prices requires extending lending facilities to shadow banking institutions and engaging in asset purchases policy. This provides a formalization of the argument that the crisis has pushed central banks to take responsibility as a liquidity back-up for the shadow banking sector that developed outside its reach, with potential benefits for financial stability (Mehrling, 2010).

**Literature Review** This work belongs to the macro-finance literature with a financial sector. Our model builds on the work of Brunnermeier and Sannikov (2014), He and Krishnamurthy (2013) and shares with these articles an incomplete financial markets structure such that the stochastic discount factor of financial intermediaries is pricing the risky assets. As in Brunnermeier and Sannikov (2016b), our model features both inside and outside money that adapts endogenously to the demand of heterogeneous agents. The main distinction between the two articles appears in the function given to money. In their work, it is held by agents as a second-best instrument to share aggregate risk. In ours, the value of money is derived from its role as the ultimate means of settlement between banks. The model in Drechsler, Savov, and Schnabl (2017) also features funding liquidity shocks affecting risk premia and asset prices through the balance sheet of intermediaries. In their model, banks always fully insured against funding risks by holding enough reserves, and monetary policy affects asset prices by varying the cost of this insurance through changes in the inflation rate. We diverge by looking at the direct effect of funding risk on risk premia and asset prices in a model where full insurance is not always feasible due to the existence of shadow banks. As in Silva (2015), we model asset purchases policy as affecting asset prices by changing the stochastic discount factor of some agent in the economy. In our model, this happens through a change in funding risk of banks instead of being the consequence of the redistribution of risks to agents without access to financial markets.
In the banking literature, Holmstrom and Tirole (1998) and Diamond and Rajan (2001, 2005) characterize optimal liquidity provision when interbank markets are affected by an aggregate liquidity shock and leads to contagion effects. By focusing on money markets and having central bank reserves as an interbank settlement asset, our work also relates to Heider, Hoerova, and Holthausen (2015) and Allen, Carletti, and Gale (2009) that show that money markets can cease to operate when credit risk is too high. Afonso and Lagos (2015) and Bech and Monnet (2016) develop over-the-counter models of the interbank market with random matching to understand its trading dynamics. Close to this article, Bianchi and Bigio (2014), Schneider and Piazzesi (2015), and Fiore, Hoerova, and Uhlig (2018) include interbank markets in macroeconomic models and study the effect of liquidity injection and lender of last resort. We extend their work by introducing a shadow banking sector, central bank asset purchases, and focusing on asset prices stability with a full-fledged consumption asset pricing model. Our paper is also linked to the literature on shadow banking: Huang (2018), Ordoñez (2018) and Plantin (2015) study the emergence of the phenomena as a consequence of regulatory arbitrage while Gennaioli, Shleifer, and Vishny (2013) and Luck and Schempp (2014) investigate the consequences for creditors of shadow banks that default. Our model is also close to Moreira and Savov (2017) as we share the view that financial fragility may arise from tightening in the collateral constraint of the shadow banking sector. We differ by characterizing shadow banks as not having access to the balance sheet of the central bank and considering different monetary policy tools through the special role of reserves as a settlement asset.

Finally, our paper relates to the macroeconomic literature that incorporates financial frictions in Neo-Keynesian models and creates a role for unconventional monetary policy as a substitute for impaired lending (Cúrdia and Woodford, 2010; Gertler and Karadi, 2011). In particular, Cúrdia and Woodford (2011) also include both central bank reserves and direct lending to non-financial companies. We depart from this literature in three ways. First, we focus on the financial stability effect of monetary policy rather than price stability. Second, in our framework, monetary policy operates by reducing liquidity risk in a context where money markets are not-functioning rather than by substituting private credit with public credit when a constraint becomes binding. Third, we discriminate between the different policies and investigate how they perform with various size of the shadow banking sector.
Figure 1: Balance Sheets of Agents in the Model. $K$ represents aggregate capital, $S$ pooled securities, $q$ the price, $N$ net worth, $D$ deposits, $M$ central bank reserves and $B$ long-term loans from the central bank to the bankers.

2 Model

The model is an infinite-horizon stochastic production economy with heterogeneous agents and financial frictions. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space that satisfies the usual conditions. Time is continuous with $t \in [0, \infty)$. The model is populated by a continuum of households, regular bankers, and shadow bankers and one central bank. Figure 1 provides a sketch of the balance sheet of these agents in equilibrium. The banking sector (shadow and regular) funds risky long-term securities holding partly through issuing instantaneous risk-free deposits to households, partly with its net-worth. The central bank operates monetary policies through its balance sheet by holding securities, lending to banks, and issuing reserves.

2.1 Environment

**Demographics** Following Drechsler, Savov, and Schnabl (2017), we assume a continuous-time overlapping generation structure à la Gârleanu and Panageas (2015) in which all agents die at rate $\kappa$ to avoid that the economy converges to a balanced growth path in which financial intermediaries own all the wealth. New agents are born at a rate $\kappa$ with
a fraction $\eta_{ss}$ as regular bankers, a fraction $\bar{\eta}_{ss}$ as shadow bankers, and $1 - \eta_{ss} - \bar{\eta}_{ss}$ as households. The wealth of all deceased agents is endowed to newly born agents equally. We denote variables specific to shadow banks with an overline and to the central bank with an underline.

**Preferences** All agents have Epstein and Zin (1989) preferences with the same parameter of risk aversion $\gamma$, intertemporal elasticity of substitution $\zeta$ and time preference $\rho$ which implicitly takes into account the probability of death $\kappa$:

$$V_t = E_t \left[ \int_t^\infty f_t du \right]$$

where $f(c_t, V_t)$ is a normalized aggregator of consumption and continuation value in each period defined as:

$$f_t = \left( \frac{1 - \gamma}{1 - 1/\zeta} \right) V_t \left[ \left( \frac{c_t}{[(1 - \gamma) V_t]^{1/(1-\gamma)}} \right)^{1-1/\zeta} - \rho \right].$$

We use this formulation in order to separate risk aversion from intertemporal elasticity of substitution. When $\gamma = 1/\zeta$, the felicity function converges to the constant relative risk aversion utility function.

**Technology** There is a positive supply of productive capital $K_t$ in the economy that yields output with constant returns to scale: $Y_t = a K_t$. All units of capital are pooled into an economy-wide diversified asset-backed security vehicle with total value $S_t$. We write the law of motion of the stock of securities as:

$$ds_t = (\Phi(\iota_t) - \delta) s_t dt + \sigma s_t dZ_t.$$ 

Where $\iota_t$ is the investment per unit of capital made by the vehicle on the behalf of the securities holders, $\delta$ is the depreciation rate and $\sigma s_t dZ_t$ is a geometric capital quality shock where $dZ_t$ is an adapted standard Brownian. The investment technology $\Phi(\cdot)$ transforms $\iota_t s_t$ units of output into $\Phi(\iota_t) s_t$ units of new securitized capital. As standard in the literature, we assume this function satisfies $\Phi(0) = 0, \Phi'(0) = 1, \Phi'(\cdot) > 0$, and $\Phi''(\cdot) < 0$. 
**Returns**  As the economy only features one aggregate stochastic process \( dZ_t \), we postulate that the stochastic law of motion of the price of a unit of securities \( q_t \) follows:

\[
\frac{dq_t}{q_t} = \mu^q_t dt + \sigma^q_t dZ_t,
\]

where \( \mu^q_t \) and \( \sigma^q_t \) are to be determined endogenously through equilibrium conditions. We write the flow of return on securities holdings as:

\[
dr^s_t = \left( \frac{a - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu^q_t + \sigma^q_t \right) dt + \left( \sigma + \sigma^q_t \right) dZ_t
\]

The drift of this process, \( \mu^s_t \), is composed of the dividend price ratio of holding a unit of securitized capital after investment plus the capital gains. This formulation assumes, without loss of generality, that the product of new investments are distributed proportionally to securities holdings. The loading factor \( \sigma^s_t \) consists in the sum of the exogenous (fundamental shock) and endogenous volatilities (corresponding response in asset prices).

**Liquidity Management**  The two types of banks are subject to idiosyncratic funding shocks. Upon the arrival of a shock, a quantity \( \sigma^d_t d_t \) of deposits in a given bank is reshuffled to another bank. This creates a funding gap for one (the deficit bank) and a funding surplus for the other (the surplus bank). As in Drechsler, Savov, and Schnabl (2017), this sequence takes place in a short period of time interval \( \Delta d \) in which loans are illiquid and can only be traded at a discount fire-sale price as compared to its fundamental value.\(^3\)

Having to fire-sale securities is costly for deficit banks. To avoid having to do so, they have the possibility to use the securities on their book as collateral to borrow from surplus banks in money markets. This process is subject to some frictions and haircuts are applied to collateral such that the amount borrowable may fall short of the funding need. In this case, shadow banks will still have to fire-sale the remaining part.

Regular banks however have two more options to mitigate this risk. First, they can hold

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\(^3\)We do not provide a micro-foundation for the cost of fire-sale but we refer the large literature in which it arises either as a consequence of shift in bargaining power under a strong selling pressure (see Duffie, 2012; Duffie, Gârleanu, and Pedersen, 2005, 2007) or asymmetry of information (see Malherbe, 2014; Wang, 1993).
central bank reserves, the ultimate interbank settlement asset, as a buffer against liquidity shocks. When the funding shock hits, reserves are immediately transferred from the deficit bank to the surplus one. Therefore, the size of the funding gap is reduced proportionally to reserves holdings. Second, they have access to the discount window facility at the central bank which makes it less costly for regular banks when they cannot access money markets. We show in Appendix C that such a problem can be written in continuous-time with the following overall transfer of wealth from a deficit to a surplus bank as:

\[
\begin{align*}
\text{shadow banks:} & \quad \theta_t \pi_t d\tilde{Z}_t = (1 - \alpha_t) \lambda_t \sigma^d w_t^d \pi_t d\tilde{Z}_t, \\
\text{regular banks:} & \quad \theta_t n_t d\tilde{Z}_t = (1 - \alpha_t) \lambda_t \max \{\sigma^d w_t^d - w_t^m, 0\} n_t d\tilde{Z}_t.
\end{align*}
\]

The variable \(\alpha_t\) is the part of the funding gap for which the deficit bank is able to cover by acquiring new fund on money markets. On this amount, the deficit bank pays a small amount \(\varepsilon\) to the surplus bank corresponding to the cost of substituting deposit funding for money market funding for the short period time \(\Delta_d\). This amount is quantitatively negligible and we simplify the model by assuming that \(\varepsilon \Delta_d \approx 0\). In order to settle the remaining amount \(1 - \alpha_t\), banks have to acquire means of payment at a higher cost by fire-selling some of their securities or accessing the discount window. This is captured by \(\lambda_t\) for regular banks and \(\bar{\lambda}\) for shadow banks. The fact that only banks have access to the discount window yields that the cost of not accessing the money market is always higher for shadow banks as compared to regular banks \(\bar{\lambda} \geq \lambda_t \geq 0\). Because everything lost by the deficit bank is gained by the surplus one, the funding risk is idiosyncratic. This idiosyncratic liquidity shock is represented by the Brownian motion \(d\tilde{Z}_t\). We assume that these transfers of wealth are instantaneous instead of lasting from \(t\) to \(t + \Delta_d\) such that we do not have to keep track of the distribution of idiosyncratic shocks.

**Central Bank**  Private agents in the economy own the central bank. To facilitate the exposition, we assume that it operates with zero net worth and instantaneously redistributes any positive (negative) realized return through a positive (negative) transfer to private

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\^4It is possible to represent this shock using either a Brownian motion or a Poisson shock. Both yield similar results, the Brownian motion yields simpler analytical results while the Poisson shock is more intuitive. In the benefit of exposition, we choose the Brownian motion. We refer to Appendix C for a discussion of the assumptions necessary for the equivalence between the two.
agents. For this reason, we scale the decision variables of the central bank by the size of the economy $q_t S_t$ and write the balance sheet identity of the central bank as:

$$\nu_t + b_t = m_t.$$ 

In this expression, $m_t = M_t / N_t$ is the supply of reserves, $\nu_t = q_t S_t / N_t$ the share of securities held by the central bank and, $b_t = B_t / N_t$ is quantity of loans from the central bank to banks. Each of these variables is scaled by the total wealth in the economy, $N_t = q_t S_t$. Considering this identity, the central bank can control two out of three of these variables. For instance, the central bank could control both the size and the composition of its balance sheet. Moreover, the central bank also sets the cost of not accessing the money market for the regular banks $\lambda_t$ as discussed previously. We therefore define the set of monetary policy decision as $\{m_t, \nu_t, \lambda_t\}$.

The distinctive role of the central bank in our economy is its capacity to issue reserves that are considered as the ultimate means of settlement in the economy. This assumption translates in our model in three ways that correspond to our three policies. First, as discussed earlier, banks can hold reserves to hedge funding shocks. Second, this is what allows the central bank to lower $\lambda_t$ in crisis: it can always grant a loan to banks after a negative shock which allows it to settle without fire-sales. Third, the central bank does not face idiosyncratic liquidity risk as will play a role when in case of direct asset purchases.

Last, we assume that the central bank may be less efficient than the private financial sector in managing securities holding and does so at a real cost of $\Gamma(\nu)$ that is a convex function of actual securities holdings. As in Cúrdia and Woodford (2010), this assumption allows us to characterize a trade-off according to which it is not trivially always optimal for the central bank to hold all the assets in the economy. It is meant to capture all potential reasons why private markets may be more efficient in managing financial assets as compared to a public bank.

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5In reality, these transfers are mediated by the fiscal authority which receives dividends from the central bank and is liable for recapitalization in case of large losses. We abstract from these concerns and assume direct transfers.
2.2 Agent’s problems

**Regular Banks** Regular bankers face a Merton’s (1969) portfolio choice problem augmented with the liquidity management component. Bankers maximize their lifetime expected recursive utility:

$$\max_{\{w^s_t, w^b_t, w^r_t, c_t\}} E_t \left[ \int_t^{\infty} e^{-\rho \tau} f(c_\tau, V_\tau) d\tau \right],$$

subject to the law of motion of wealth:

$$dn_t = \left( w^s_t \mu^s_t + w^b_t r^b_t + w^m_t r^m_t - w^d_t r^d_t - c_t + \mu^d_t \right) n_t dt + (w^s_t \sigma^s_t + \sigma^d_t) n_t dZ_t$$

$$+ (1 - \alpha_t) \lambda_t \max \left\{ \sigma^d w^d_t - w^m_t, 0 \right\} n_t d\tilde{Z}_t,$$  \hspace{1cm} (2)

and the balance sheet constraint:

$$w^s_t + w^b_t + w^m_t = 1 + w^d_t.$$ \hspace{1cm} (3)

Regular bankers face a portfolio choice problem with four different assets: securities portfolio weight $w^s_t$, interbank lending with portfolio weight $w^b_t$, central bank reserves portfolio weight $w^m_t$, and deposits portfolio weight $w^d_t$. In equation (2), $r^b_t$ is the interest rate on interbank lending, $r^m_t$ the interest rate paid by the central bank on its reserves, and $r^d_t$ the interest rate on deposits. Banks also choose their consumption rate $c_t$. Bankers receive a flow of transfers per unit of wealth of $d\tau_t = \mu^d_t dt + \sigma^d_t dZ_t$ from the central bank. The last term of equation (2) reflects the effect of the liquidity management problem of the regular banks on the flow of returns as described previously.

**Shadow Banks** Shadow bankers face a similar problem as banks except for the difference in their access to the central bank balance sheet:

$$\max_{\{\pi^s_t, \pi^b_t, \pi^d_t, \tau_t\}} E_t \left[ \int_t^{\infty} e^{-\rho \tau} f(\pi_t, \bar{V}_\tau) d\tau \right],$$

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subject to the law of motion of wealth:

\[ d\tilde{n}_t = \left( w^s_t \mu^s_t + w^b_t r^b_t - w^d_t r^d_t - \tilde{c}_t + \tilde{\mu}_t \right) \tilde{n}_t dt + \left( w^s_t \sigma^s_t + \sigma^\tau_t \right) \tilde{n}_t d\tilde{Z}_t \]

and the balance sheet constraint:

\[ w^s_t + w^b_t = 1 + w^d_t. \]

The interpretation of the variables, now overlined to denote shadow bankers, is the same as for regular bankers.

**Households** Households maximize their life-time utility function subject to the additional assumption that they can only invest in bank deposits:

\[ \max_{\{c^h_t\}_{t=0}} E_t \left[ \int_t^\infty e^{-\rho^* \tau} f(c^h_t, V^h_t) d\tau \right], \]

subject to the law of motion of wealth:

\[ dn^h_t = (r^d_t - c^h_t) n^h_t dt, \]

where the \( h \) index refers to households.

**Equilibrium Definition**

**Definition 1** (Sequential Equilibrium) Given an initial allocation of all asset variables at \( t = 0 \), monetary policy decisions \( \{m_t, \nu_t, \lambda_t : t \geq 0\} \), and transfer rules \( \{\sigma^s_t, \sigma^\tau_t, \mu^s_t, \mu^\tau_t : t \geq 0\} \), a sequential equilibrium is a set of adapted stochastic processes for (i) prices \( \{q_t, r^b_t, r^m_t, r^d_t : t \geq 0\} \), (ii) individual controls for regular bankers \( \{c_t, w^s_t, w^m_t, w^b_t, w^d_t : t \geq 0\} \), shadow bankers \( \{\bar{c}_t, \bar{w}^s_t, \bar{w}^m_t, \bar{w}^b_t : t \geq 0\} \), and for households \( \{c^h_t : t \geq 0\} \), (iii) security issuance rate \( \{\iota_t : t \geq 0\} \), (iv) aggregate security stock \( \{S_t : t \geq 0\} \), and (v) agents’ net worth \( \{n_t, \bar{n}_t, n^h_t : t \geq 0\} \) such that:

1. Agents solve their respective problems defined in equations (1), (4), and (7).
2. Markets for securities, interbank lending, reserves, and consumption goods clear:

(a) securities: \[ \int I_1 w^s_t n^s_t \, di + \int J_1 w^s_t \pi_t \, dj = (1 - \nu_t)S_t \]

(b) interbank lending: \[ \int I_1 w^b_t n^b_t \, di + \int J_1 w^b_t \pi_t \, dj = b_t q_t S_t, \]

(c) reserves: \[ \int I_1 w^m_t n^m_t \, di = m_t q_t S_t, \]

(d) output: \[ \int I_1 c_t n_t \, di + \int J_1 \pi_t \pi_t \, dj + \int H_h c^h_t n^h_t \, dh = (a - \iota - \Gamma(\nu))S_t. \]

2.3 Discussion of Assumptions

**Market Segmentation** We view the market segmentation hypothesis as a parsimonious way of writing down a model where there is some constraint on the risk sharing between the two sectors that is binding when there is a crisis such that the stochastic discount factor of intermediaries is pricing the risky assets in the economy (a feature for which there is strong empirical support; see for instance Adrian, Etula, and Muir, 2014; He, Kelly, and Manela, 2017). We refer to Brunnermeier and Sannikov (2016a) and He and Krishnamurthy (2013) for a micro-foundation of such a constraint originating from agency frictions forcing bankers to keep some skin in the game when holding risky assets and preventing optimal risk sharing. We could allow for the constraint to be only occasionally binding without affecting our main results as we are focussed on states where this constraint is tight.

**Shadow Banks** As in Pozsar, Adrian, Ashcraft, and Boesky (2012) and Adrian and Ashcraft (2012), we see the lack of access to public sources of liquidity as an essential distinction between the shadow and traditional banking sector. In order to be able to focus on this aspect, we model shadow banks as exactly similar to traditional banks in all other accounts. This assumption corresponds to two existing institutional features. First, in most countries, only institutions licensed as banks (in the US, called depository institutions) have an account at the central bank and, hence, can hold reserves. Second, access to lender of last resort facilities (such as the Fed discount window) is usually also restricted to the same set of institutions. In this setting, we interpret a policy that extends the lender of last resort function to a larger set of institutions, such as the creation of the...
Primary Dealer Credit Facility or Central Bank Swaps\(^6\) lines in 2008, as transforming some shadow banks into traditional banks.

**Discount Window Policy** We model the discount window policy by having the central bank affecting the overall cost of being illiquid for banks rather than the discount window rate. The reason behind this modeling choice is that we see the discount window policy as a multiple dimension object. In reality, various variables affect the cost of a liquidity shortage for a traditional bank. For instance, the literature has documented a strong negative stigma in accessing the discount window at the Fed, especially during a financial crisis (Armantier, Ghysels, Sarkar, and Shrader, 2015). In an attempt to reduce the stigma of borrowing funds at the discount window, the Fed introduced a new lending facility for banks, the Term Auction Facility (TAF), in 2007\(^7\). Moreover, discount window loans are, in practice, collateralized. This means that for the policy to be effective, the central bank needs to be less restrictive than markets in the set of eligible collateral. By accepting more or less securities as collateral, the central bank may have significant impact on the funding risks of banks. This channel has been particularly important in Europe when Treasuries of peripheric countries were applied sizable haircuts (Bindseil, 2013). We capture these different dimensions in which the central bank can affect the availability and cost of discount window policy through the variable \(\lambda_t\).

**Transfers Rules** Our assumption regarding the transfer rules is set-up in order to shut down any redistribution channel of monetary policy. As we will show later, with this rule, asset purchase policies are Wallace neutral in absence of liquidity risk. We do so for two reasons. First, distributional effects of monetary policy in this class of model have already been studied extensively (Brunnermeier and Sannikov, 2014; He and Krishnamurthy, 2013; Silva, 2015). Second, this allows us to focus on the liquidity risk channel of monetary policy which is the focus of this article.

\(^6\)A currency swap line is an agreement between two central banks to exchange currencies. They allow a foreign central bank to provide (dollar) funding to its domestic banks in case of liquidity stress in (dollar) money markets.

\(^7\)TAF auctions were designed such that the amount of funding available is announced in advance, which made it less likely that market participants would infer that borrowing institutions had an immediate need for funds (Carlson and Rose, 2017)
2.4 Solving

Each agent’s optimal decision depends on the functioning of money markets, monetary policy, and equilibrium market prices. The homotheticity of Epstein-Zin preferences generates optimal strategies that are linear in the net worth of a given agent. Therefore, the distribution of net worth within each sector does not affect the equilibrium. We characterize the equilibrium as in Brunnermeier and Sannikov (2014) and Di Tella (2017) by using a recursive formulation of the problem, taking into account the scale invariance of the model which allows to abstract from the level of aggregate capital stock. We guess and verify that the value function of each agent has the following power form:

\[ V(n_t) = \frac{(\xi_t n_t)^{1-\gamma}}{1-\gamma}, \quad V(\pi_t) = \frac{(\xi_t \pi_t)^{1-\gamma}}{1-\gamma}, \quad V^h(n^h_t) = \frac{(\xi^h_t n^h_t)^{1-\gamma}}{1-\gamma}. \]

for some stochastic processes \( \{\xi_t, \pi_t, \xi^h_t\} \) capturing time variations in the set of investment opportunities for a given type of agent. A unit of net worth has a higher value for a regular bank, a shadow bank, or a household in states where \( \xi_t, \pi_t \) or \( \xi^h_t \) are respectively high. Without loss of generality, we postulate that the law of motion for these wealth multipliers follows an Ito Process:

\[
\frac{d\xi_t}{\xi_t} = \mu_{\xi} dt + \sigma_{\xi} dZ_t, \quad \frac{d\pi_t}{\pi_t} = \mu_{\pi} dt + \sigma_{\pi} dZ_t, \quad \frac{d\xi^h_t}{\xi^h_t} = \mu_{\xi^h} dt + \sigma_{\xi^h} dZ_t.
\]

**Recursive Formulation**  As a consequence of the homotheticity of preferences and linearity of technology, all agents of a same type choose the same set of control variables when stated in proportion of their net worth. Hence, we only have to track the distribution of wealth between types and not within types. The two state variables of the economy are the share of wealth in the hands of the regular and shadow banking sectors:

\[ \eta_t \equiv \frac{n_t}{n_t + \pi_t + n^h_t}, \quad \overline{\eta}_t \equiv \frac{\pi_t}{n_t + \pi_t + n^h_t}, \]

where the total net worth in the economy is given by \( n_t + \pi_t + n^h_t = q_t S_t \). From here on, we characterise the economy as a recursive Markov equilibrium.

**Definition 2** (Markov Equilibrium) A Markov equilibrium in \((\eta_t, \overline{\eta}_t)\) is a set of functions \( f_t = f(\eta_t, \overline{\eta}_t) \) for (i) prices \( \{q_t, r^d_t, r^m_t, r^h_t\} \), (ii) individual controls for regular bankers.
\{c_t, w_t^s, w_t^b, w_t^d\}, shadow bankers \{\bar{c}_t, \bar{w}_t^s, \bar{w}_t^d\}, and for households \{c^h_t\}, (iii) security issuance rate \{\nu_t\}, (iv) monetary policy functions \{m_t, \nu_t, \xi_t\} and transfer rules \{\sigma^t_t, \bar{\sigma}^t_t, \mu^t_t, \bar{\mu}^t_t\} such that:

1. Wealth multipliers \{\xi_t, \bar{\xi}^s_t, \bar{\xi}^h_t\} solve their respective Hamilton-Jacobi-Bellman equations with optimal controls (ii), given prices (i), monetary policy and transfer policy (iv).

2. Markets for securities, interbank lending, reserves, and consumption goods clear:

   (a) securities: \[w_t^s \eta_t + \bar{w}_t^s \bar{\eta}_t + \nu_t = 1\]
   (b) interbank lending: \[w_t^b \eta_t + \bar{w}_t^b \bar{\eta}_t = \bar{\nu}_t,\]
   (c) reserves: \[w_t^m \eta_t = m_t,\]
   (e) output: \[c_t \eta_t + \bar{c}_t \bar{\eta}_t = (a - \iota_t - \Gamma(\nu_t))/q_t.\]

3. Monetary policy \(m_t, \nu_t, \xi_t\) are set only as functions of the state variables.

4. Transfers rules \(\sigma^t_t, \bar{\sigma}^t_t, \mu^t_t, \bar{\mu}^t_t\) are given by:

   \[\sigma^t_t = \bar{\sigma}^t_t = \frac{\nu_t}{\eta_t + \bar{\eta}_t} \sigma^s_t,\]
   \[\mu^t_t \eta_t = \frac{\eta_t}{\eta_t + \bar{\eta}_t} (\mu^s_t - r^d_t) \nu_t + (r^b_t - r^m_t) \bar{\nu}_t - (r^m_t - r^d_t) m_t,\]
   \[\bar{\mu}^t_t \bar{\eta}_t = \frac{\bar{\eta}_t}{\eta_t + \bar{\eta}_t} (\mu^s_t - r^d_t) \nu_t.\]

5. The laws of motion for the state variables \(\eta_t, \bar{\eta}_t\) are consistent with equilibrium functions and demographics.

**First Order Conditions** The optimality conditions for control variable are derived in the appendix by writing the stationary Hamilton-Jacobi-Bellman equations. With a little bit of algebra, we can write these conditions for securities holdings as:

regular banks: \[\mu^s_t - r^b_t = \gamma(w^s_t \sigma^s_t + \bar{\sigma}^s_t) \sigma^s_t - (1 - \gamma) \sigma^s_t \sigma^s_t,\]
shadow banks: \[\mu^s_t - r^b_t = \gamma(w^s_t \sigma^s_t + \bar{\sigma}^s_t) \sigma^s_t - (1 - \gamma) \bar{\sigma}^s_t \sigma^s_t.\]
The excess return on the risky asset must be equal to minus the covariance between the return process and the stochastic discount factor. More precisely, excess returns compensate for taking exposure in two types of risks. The first term takes into account variations in marginal utility originating purely from the additional wealth volatility. The second term corresponds to the compensation for correlated changes in the set of investment opportunities. So far, these conditions correspond to the traditional portfolio problem. We can similarly derive the first order conditions for the portfolio weights on deposit holdings:

\[
\begin{align*}
\text{regular banks:} & \quad r^b_t - r^d_t = \gamma (1 - \alpha_t)^2 \lambda_t^2 \max\{\sigma^d w^d - w^m, 0\} \sigma^d \\
\text{shadow banks:} & \quad r^b_t - r^d_t = \gamma (1 - \alpha_t)^2 \lambda^w w^d (\sigma^d)^2
\end{align*}
\]

(10) (11)

From the point of view of banks, issuing short-term deposits is risky as it creates an exposure to funding shocks. As deposits are a liability of banks, this additional exposure needs to be compensated by a negative premium with respect to the risk free interbank market rate \( r^b_t \). For both types of banks, this negative premium is equal to the marginal cost of the corresponding increase in liquidity risk. This effect is increasing in money markets frictions \( \alpha_t \) and disappears when money markets are working perfectly \( (\alpha_t = 1) \).

We can derive a similar condition for reserves holdings from regular banks:

\[
\begin{align*}
r^b_t - r^m_t = \gamma (1 - \alpha_t)^2 \lambda_t^2 \max\{\sigma^d w^d - w^m, 0\}.
\end{align*}
\]

(12)

This equation looks similar to the one for deposits but has an opposite interpretation. In this case, central bank reserves are an asset from the perspective of banks and holding it reduces the effect of funding shocks on wealth. Therefore reserves also requires a negative premium with respect to the risk free interbank market rate \( r^b_t \) (the marginal cost) that is equal to the marginal reduction in the impact of the funding shock (the marginal benefit).

As all agents have the same preferences, their optimal consumption choices are given by:

\[
\begin{align*}
c_t &= \xi_t^{1-\zeta} \\
\bar{c}_t &= \xi_t^{1-\zeta} \\
c^h_t &= (\xi_t^{h})^{1-\zeta}
\end{align*}
\]

(13) (14) (15)

Agents’ consumption rates depends on their set of investment opportunities and their intertemporal elasticity of substitution parameter \( \zeta \). When \( \zeta > 1 \), the substitution dominates the wealth effect and agents react to an improvement of their set of investment opportunities by decreasing consumption. The reverse holds when \( \zeta < 1 \) and both effects cancel out when \( \zeta = 1 \).
3 Static Results

In this section, we first study how money market frictions affect the economy. In particular, we show that an increase in money market frictions results in a drop in asset prices as higher funding liquidity risk impacts the stochastic discount factor of banks. We then investigate how the different types of monetary policy may affect allocations and prices under various liquidity regimes. We show how the different policies may break Wallace’s (1981) neutrality result in the presence of impaired money market. We then show that, in the presence of a large shadow banking sector, liquidity injections and better discount window conditions may not be sufficient to alleviate funding risk and asset prices stabilization may require asset purchase policy in order to affect the whole banking sector.

To facilitate the exposition, we make a technical assumption to shut down the distribution of wealth as state variables as it is inessential for the results. More explicitly, assume that the death rate $\kappa \to \infty$ such that $\eta_t = \eta_{ss} \equiv \eta$ and $\bar{\eta}_t = \bar{\eta}_{ss} \equiv \bar{\eta}$ and, consequently, drop the subscript $t$ for all variables. We release this assumption in the next section and show that our results are not impacted.

3.1 Benchmark Without Liquidity Risk

Without liquidity risk, i.e. in a world where there are no money market frictions ($\alpha = 1$), the model yields the following solution:

**Lemma 3** (Prices without Liquidity Risk) *In the absence of money market frictions ($\alpha = 1$), equilibrium prices along the balanced growth path are given by:

$$q = \frac{a - \iota}{\rho - (1 - \zeta^{-1}) \left( \Phi(\iota) - \frac{\gamma}{2} \frac{\sigma^2}{\eta + \bar{\eta}} \right)},$$

$$r_m = r^b = r^d = \rho - \zeta^{-1} \Phi(\iota) + (1 - \zeta^{-1}) \frac{\gamma}{2} \frac{\sigma^2}{\eta + \bar{\eta}}.$$

*All proofs are relegated to the Appendix.*

This benchmark corresponds to the traditional consumption-based asset pricing equa-

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$^8$We also assume that agents value the bequest they leave exactly such that $\rho$ remains unaffected by the change in $\kappa$ as a technical assumption.
tion adjusted for recursive preferences and the wealth share of the aggregate banking sector \( \eta + \bar{\eta} \). As intermediaries are the only agents that can bear fundamental risk, the precautionary motives take into account that banks are levered and have to bear a risk of \( \gamma \sigma^2/ (\eta + \bar{\eta}) \) per unit of wealth. The rest of the equations is standard. The price of securities is the discounted value of the flow of future dividends \( a \). When the intertemporal elasticity of substitution is above one, \( \zeta > 1 \), the substitution effect dominates such that an increase in the drift of the capital accumulation process \( \Phi(\iota) \) results in higher prices while an increase in uncertainty \( \sigma^2 / (\eta + \bar{\eta}) \) decreases asset prices. We focus on this case as it is standard in the macro-finance literature. For completeness, note that when the converse holds, \( \zeta < 1 \), the wealth effect dominates such that these relationships go in the opposite direction. The deposit rate also depends on the intertemporal elasticity of substitution. In particular, when the substitution effect dominates, an increase in uncertainty or decrease in the banking sector relative wealth yields a reduction in the rate on deposits.

**Proposition 4 (Neutrality of Monetary Policy Instruments without Liquidity Risk)** In the absence of money market frictions (\( \alpha = 1 \)), any change in the monetary policy decision set \( \{m, \nu, \lambda\} \) has no effect on any equilibrium variables.

This result is straightforward for both liquidity injection policies (a change in \( m \)) and discount window policy (a change in \( \lambda \)) as the only equation in which these variables appear are the first order condition for deposits and reserves of banks and is always scaled by \( 1 - \alpha = 0 \). In other words, the only effect of these policies is to lower the liquidity risk of banks. Yet, when money markets functioning perfectly, this liquidity risk is already null such that any liquidity or discount window policy change is inconsequential. The reason behind the neutrality of asset purchases policy is different. Whenever the central bank purchases risky securities, banks keep their exposure to the underlying fundamental risk through the transfer functions. This can be seen by first noting that market clearing conditions and the symmetry between two types of banks absent liquidity risk implies that:

\[
\begin{align*}
  w &= \frac{\eta}{\eta + \bar{\eta}} (1 - \nu), \\
  \bar{w} &= \frac{\bar{\eta}}{\eta + \bar{\eta}} (1 - \nu).
\end{align*}
\]

After substituting for both portfolio weights and transfer rules, we can rewrite the asset
pricing equations for optimal risky securities holdings as:

\[ \mu^s - r^d = \gamma \left( \frac{\eta}{\eta + \eta} (1 - \nu) \sigma^s + \frac{\eta}{\eta + \eta} \nu \sigma^s \right) \sigma^s; \]
\[ \mu^s - r^d = \gamma \left( \frac{\eta}{\eta + \eta} (1 - \nu) \sigma^s + \frac{\eta}{\eta + \eta} \nu \sigma^s \right) \sigma^s; \]

in which central bank holdings of risky securities \( \nu \) cancels out. As agents understand this exposure they adjust their demand for securities exactly such that the aggregate demand remains unaffected. These results are simply a restatement of the seminal Wallace’s (1981) neutrality result in the risk space.

### 3.2 Money Markets Frictions

In this subsection, we focus on equilibrium with money market frictions \( \alpha < 1 \) but without monetary policy \( \nu = m = 0 \) and \( \lambda = \lambda \). For simplicity, we also assume that \( \sigma^d = 1 \) and use the degree of freedom that we have in \( \lambda \) and \( \alpha \) to vary the scale of the funding shock. We first combine the first order conditions for securities and deposits for the two banks by substituting out the risk-free interbank money market rate \( r^b \):

\[ \mu^s - r^d = \gamma w^s \sigma^2 - \gamma (1 - \alpha)^2 \lambda^2 w^d. \]  

Equation (16) alread shows that banks take into account that they need to raise deposits that generates liquidity risk when choosing their demand for securities. Thus, they trade-off an increase in both fundamental and funding liquidity risk for excess returns.

We can now write the closed form solution of the model in the case where there is liquidity risk and no monetary policy.

**Proposition 5 (Prices with Liquidity Risk and No Central Bank)** In an economy without asset purchase \( \nu = 0 \), without reserves \( m = 0 \), and without discount window facility \( \lambda = \lambda \), equilibrium securities prices along the balanced growth path are given by:

\[ q = \frac{a - \iota}{\rho - (1 - \zeta^{-1}) \left( \Phi - \frac{\gamma}{2} (\Omega \sigma^2 + \Psi) \right)}, \]  

20
The figure displays how securities prices and net interest margin react to a change in money market frictions as a function of the wealth of the banking sector: benchmark with $\alpha = 1$ in black, $\alpha = 0.7$ in blue, and $\alpha = 0.05$ in red. The other parameters are set according to: $\alpha = 0.05$, $\rho=0.03$, $\zeta^{-1}=0.7$, $\Phi=0.02$, $\gamma = 1.1$, $\sigma=0.03$, $\lambda = 0.5$.

Where

$$\Omega = \frac{1}{\eta + \bar{\eta}}, \quad \Psi = (1 - \alpha)^2 \lambda^2 \frac{(1 - \eta - \bar{\eta})^2}{\eta + \bar{\eta}}.$$

When the substitution effect dominates, an increase in funding risks in the economy (due to higher money market frictions) leads to a decrease in asset prices. This can be seen in the extra-term $\Psi$ of equation (17) as compared to the benchmark without liquidity risk. Idiosyncratic funding liquidity risk is part of the asset price as it is undiversifiable and, therefore, part of the pricing kernel of financial intermediaries. The function $\Psi$ is scaling the funding risk to the equilibrium leverage of the financial sector. Note that when banks hold all the wealth in the economy ($\eta + \bar{\eta} = 1$), banks have no leverage and $\Psi = 0$ such that there is no funding risk component in the asset pricing equation. In figure 2, we compare equilibrium for different levels of liquidity risk as a function of $\eta + \bar{\eta}$. For a higher level of liquidity risk due to poorer money market conditions, asset prices are lower and the net interest margin is higher.

3.3 Monetary Policy Instruments

In this subsection, we decompose the impact and limitations of the different monetary policy instruments. First, we clarify the position of interest on reserves in our framework
and why it is not included in the set of monetary policy instruments. Then, we look at liquidity injection, lender of last resort, and asset purchase policies. We show how both liquidity injection and discount window policies are limited as they cannot reach the shadow banking sector while asset purchases gets in all the cracks by reducing funding liquidity risk through a general equilibrium effect.

**Interest Paid on Reserves** In setting up our model, we have not incorporated the interest paid on reserves (IOR) in the toolbox of the central bank but rather as an equilibrium outcome. Today, most central banks decide on and frequently adjusts their IOR to economic conditions as a monetary policy tool\(^9\). In order to show that the model is consistent with IOR being a monetary policy variable in a nominal world, let’s define the nominal interest on reserves \(i^m = r^m + \pi_t\) where \(\pi_t\) is the inflation rate.\(^10\) We can combine this equation with the asset pricing condition for reserve (12) to find:

\[
\pi = i^m + \gamma(1 - \alpha)^2 \lambda^2 \max\{\sigma^d w^d - m, 0\} - r^b.
\]

Inflation is uniquely determined by this equation as the deviation between the nominal and real interest rates prevailing in money markets. The nominal money market rate is composed of two terms: the nominal interest on reserves and the real money market spread determined by equation (12). The central bank can directly affect this spread as it is the supplier of reserves. Thus, the central bank can affect inflation with two different policy tools: the nominal interest rate on reserves \(i^m\) and the supply of reserves \(m^\).\(^11\) This equation corresponds to the classic Fischer equation and since the model does not feature

---

\(^9\)For instance, the Fed received legal authority to pay interest on reserves in 2008. Even before this period, the interest paid by the Fed was constrained to be a nominal zero. This would also be at odds with our assumption that \(r^m\) is market determined.

\(^10\)The nominal world is defined by \(P_t\) being the price the numeraire output and assuming that prices changes deterministically such that \(dP_t/P_t = \pi_t dt\).

\(^11\)This result is consistent with observed heterogeneity in the implementation practices of central banks. For example, until 2011, the Federal Reserve was not providing a deposit facility to excess reserves, implicitly setting the interest on excess reserves to zero. Every adjustment in the monetary policy stance was, therefore, taking place as a shift in the spread implemented by daily adjustments in the supply of excess reserves. Conversely, since its establishment, the European Central Bank has been following a symmetrical corridor operational framework. Under this regime, the ECB sets the bounds of the corridor at a fixed 200 basis points spread and adjusts the reserve supply in order for the spread to clear halfways. In this case, the ECB implements its monetary policy stance effectively by shifting the interest on excess reserves \(i^m\t\) (deposited at the ECB) rather than moving the spread \(r^b - r^m\).
nominal rigidities, the relationship between inflation and the risk-free rate is positive. As price stability is not the focus of this article, we abstract from these considerations with Assumption 6 that is a sufficient condition for a model where the central bank controls IOR to reduce to the model described in section 2.

**Assumption 6 (Separation Principle)** The central bank stabilizes inflation to zero by pinning down the nominal interest rate paid on reserves:

\[ i^m = i^b - \gamma (1 - \alpha)^2 \lambda^2 \max\{\sigma^d w^d - m, 0\} \text{ such that } \pi = 0. \]

Assumption 6 is has an intuitive interpretation as it reflects the practice in many central banks during the Great Recession, referred to as the *separation principle*, according to which the degree of freedom in the monetary policy toolbox allows to have the interest on reserves focused on maintaining price stability while the stock of reserves can be adjusted independently to alleviate liquidity stresses in the interbank market (e.g., Clerc and Bordes, 2010).

**Liquidity Injections** As regular banks hold reserves in order to hedge against funding liquidity shocks, an increase in the supply of reserves can affect asset prices whenever money markets are functioning imperfectly.

**Proposition 7 (Prices with Positive Supply of Reserves)** In an economy without asset purchase \( \nu = 0 \) and without a discount window facility \( \lambda = \bar{\lambda} \), equilibrium securities prices along the balanced growth path are given by:

\[
q = \frac{a - \iota}{\rho - (1 - \zeta^{-1}) \left( \Phi - \frac{1}{2} (\Omega(m)\sigma^2 + \Psi(m)) \right)}
\]

(18)
where

\[ \Omega(m) = 1 + \frac{m^2 (1 - \alpha)^2 \lambda^2 \eta}{\sigma^2 + (1 - \alpha)^2 \lambda^2 \eta} \left( \frac{1}{\eta + \bar{\eta}} \right), \]

\[ \Psi(m) = \begin{cases} 
(1 - \alpha)^2 \lambda^2 \frac{(1 - \eta - \bar{\eta} - m)^2}{\eta + \bar{\eta}} & \text{if } m \leq m^*, \\
(1 - \alpha)^2 \lambda^2 \frac{(1 - \eta - \bar{\eta} - m^*)^2}{\eta + \bar{\eta}} & \text{otherwise,} 
\end{cases} \]

\[ m^* = (1 - \eta - \bar{\eta}) \frac{\sigma^2 + (1 - \alpha)^2 \lambda^2}{\sigma^2 + (1 - \alpha)^2 \lambda^2 + \frac{\eta}{\eta^2}}. \]

The supply of central bank reserves enters in two ways in the asset pricing equation (18). First, through the term \( \Psi(m) \), the stock of funding liquidity risk, an increase in money supply \( m \) has a positive impact on asset prices until reaching \( m^* \), which corresponds to the reserve satiation threshold of regular banks. As the central bank increases the supply of reserves, banks have to face lower liquidity risk. This positive effect is dampened through the term \( \Omega(m) \). The intuition is that, as funding liquidity risk becomes lower for regular banks as compared to shadow banks, the former type of bank start to hold a large share of the existing stock of securities. The distribution of fundamental risk \( \sigma^2 \) becomes asymmetrical and introduce an inefficiency as compared to what is optimal which has a negative impact on securities prices. This dampening effect is proportional to the relative size of the shadow banking sector \( \bar{\eta}/\eta \).

Figure 3 illustrates how the size of the shadow banking sector is playing a role in determining where the liquidity satiation threshold is located. The black line represents the benchmark economy without liquidity risk. The red line shows how the supply of reserves affect the variables when there are only traditional banks. In this case, the central bank is able to inject enough liquidity to make sure that regular banks are fully satiated. At this point \( m^* \), there is no more liquidity risk in the economy and asset prices are equal to the benchmark. When the shadow banking sector is large (blue line), traditional banks may be liquidity-satiated while there is still a significant amount of funding liquidity risk in the economy and asset prices are below the benchmark level.
Figure 3: The figure displays securities prices, stocks of liquidity risk, and the dampening effect as a function of the supply of reserves: benchmark without funding liquidity risk in black ($\alpha = 1$); without shadow banks in red ($\alpha = 0.7$, $\eta = 0$, $\eta = 0.5$); with a large shadow banking sector in blue ($\alpha = 0.7$, $\eta = 0.35$, $\eta = 0.15$). The other parameters are set according to: $a=0.05$, $\rho=0.03$, $\zeta^{-1}=0.7$, $\phi=0.02$, $\gamma=1.1$, $\sigma=0.03$, $\lambda = 0.5$.

**Discount Window**  By lowering the cost of using the discount window rate (or facilitating its usage), the central bank reduces the cost of being illiquid for banks $\lambda$. In doing so, the central bank affect positively equilibrium prices.

**Proposition 8 (Prices with Discount Window)** In an economy without asset purchase $\nu = 0$ and without liquidity injections $m = 0$, equilibrium securities prices along the balanced growth path are given by:

$$q = \frac{a - \iota}{\rho - (1 - \zeta^{-1}) \left( \Phi - \frac{\eta}{2} (\Omega(\lambda)\sigma^2 + \Psi(\lambda)) \right)}$$

where

$$\Omega(\lambda) = \frac{\sigma^2 + \frac{1}{2}(\theta^2 + \bar{\theta}^2)}{(\sigma^2 + \theta^2)\bar{\eta} + (\sigma^2 + \bar{\theta}^2)\eta},$$

$$\Psi(\lambda) = \eta \theta^2 + \bar{\eta} \bar{\theta}^2 + \frac{\theta^2 \bar{\theta}^2 + \sigma^2 (\theta^2 + \bar{\theta}^2) - (\bar{\theta}^2 - \theta^2)^2 \eta \bar{\eta} - 2\theta^2 (\sigma^2 + \bar{\theta}^2) \eta - 2\bar{\theta}^2 (\sigma^2 + \theta^2) \bar{\eta}}{(\sigma^2 + \theta^2)\bar{\eta} + (\sigma^2 + \bar{\theta}^2)\eta}.$$

Although these equations are different from the ones for the liquidity injection policy, they have a close interpretation. The term $\Psi(\lambda)$ accounts for the direct reduction of funding risks for traditional banks when $\theta(\lambda)$ is lowered. At the extreme, if the central bank does
not set a discount window such that \( \lambda = \lambda_c \), then \( q \) reverts back to equation (17). On the other hand, if the discount window such that there is no more liquidity risk for traditional banks \( \lambda = 0 \), then \( \Psi(0) \) reduces to:

\[
\Psi(0) = \theta^2 \left( \frac{\eta + \sigma^2}{\sigma^2 \eta + (\sigma^2 + \theta^2) \eta} \right) \geq \frac{\theta^2 (1 - \eta - \bar{\eta})^2}{\eta + \bar{\eta}}. \tag{20}
\]

As can be seen from equation (20), shutting down the liquidity risk in the traditional banking sector through better lender of last resort conditions may not be sufficient to push asset prices back to the benchmark level without liquidity risk. The variable \( \Omega(\lambda) \); here again, takes into account the skewness in the holdings of fundamental risk between traditional and shadow banks that is introduced by the liquidity risk advantage that central bank is providing to banks. When liquidity risk is symmetric (i.e. absent discount window policy \( \lambda = \lambda_c \)), the dampening effect \( \Omega \) converges to its value in the benchmark case \( 1/(\eta + \bar{\eta}) \).

**Asset Purchases**  The last type of policy we consider is the direct purchase of securities by the central bank. In an economy where money markets function imperfectly (\( \alpha < 1 \)), asset purchases have a positive impact on asset prices.

**Proposition 9 (Prices with Central Bank Securities Holdings)**  *In an economy without a discount window facility \( \lambda = \lambda_c \), equilibrium securities prices along the balanced growth path are given by:

\[
q = \frac{a - \iota - \Gamma(\nu)}{\rho - (1 - \zeta^{-1}) (\Phi - \frac{\gamma}{2} (\Omega \sigma^2 + \Psi(\nu)))}, \tag{21}
\]

where

\[
\Omega = \frac{1}{\eta + \bar{\eta}},
\]

\[
\Psi(\nu) = \begin{cases} 
(1 - \alpha)^2 \lambda^2 \frac{(1 - \eta - \bar{\eta} - \nu)^2}{\eta + \bar{\eta}}, & \text{if } \nu \leq 1 - \eta - \bar{\eta} \\
0, & \text{otherwise.}
\end{cases}
\]

Central bank securities holdings affect the economy in two opposite ways through two different terms in equation (21). First, a purchase of securities has positive effect through
Figure 4: The figure displays securities prices, stocks of liquidity risk, and the convex cost of central bank management as a function of central bank share of securities holdings: benchmark without funding liquidity risk in black ($\alpha = 1$); without convex cost in red ($\Gamma(\nu) = 0$, $\alpha = 0.7$); with a quadratic convex cost in blue ($\Gamma(\nu) = 0.015 \times \nu^2$, $\alpha = 0.7$). The other parameters are set according to: $a=0.05$, $\rho=0.03$, $\zeta^{-1}=0.7$, $\Phi=0.02$, $\gamma=1.1$, $\sigma=0.3$, $\lambda=0.5$, $\eta=0.2$ and, $\eta=0.2$.

$\Psi(\nu)$. When the central bank buys securities, it removes a stock of idiosyncratic funding liquidity risk from banks’ balance sheets. Because the central bank does not face funding liquidity risk, these risks are extracted from the economy and, unlike fundamental risk, are not passed on to banks through future transfers. This results, in general equilibrium, in a lower stock of funding liquidity risk (see equation (9)). The scaling factor of liquidity risk $\Psi(\nu)$ is a negative function of $\nu$ up to the point where the central bank has bought all securities. Importantly, these securities previously funded by short-term deposits are replaced with reserves on the balance sheets of banks. The asset purchase policy also has a negative impact on asset prices through the real resource convex cost $\Gamma(\nu)$. The overall impact on securities price is a quantitative question that depends on the balance between these two forces as is illustrated in figure 4.

4 Dynamic Results

In this section, we endogenize the frictions in the money market by modeling explicitly the haircut necessary to secure trades given the volatility of assets. Then we show that the resulting collateral spiral strongly amplifies the drop in asset prices subsequent to a series of negative shocks. Finally, we investigate, in the fully dynamic setting, how the different monetary policies may partially counteract the collateral spiral.
4.1 Numerical Procedure and Parametrization

We solve numerically for the global solution of the model, that is, the mapping from the pair of state variables \( \{ \eta_t, \bar{\eta}_t \} \) to all equilibrium variables. The numerical procedure follows the finite-difference methodology introduced in Brunnermeier and Sannikov (2016a) and extended for two state variables in d’Avernas and Vandeweyer (2017). The procedure decomposes the approximation scheme in two separate parts. We solve for the wealth multiplier \( \xi(\eta_t, \bar{\eta}_t) \), \( \bar{\xi}(\eta_t, \bar{\eta}_t) \) and \( \xi^h(\eta_t, \bar{\eta}_t) \) backward in time by using an implicit Euler method. We evaluate the finite difference approximation of the derivative terms in the right direction to preserve the numerical stability of the scheme following Barles and Souganidis (1990). In between these time steps, we solve for the system of equations using the Newton-Raphson method.

4.2 Endogeneous Collateral Constraint

Until this point, we have treated the proportion of available collateral \( \alpha \) as a parameter. In reality, this variable varies through time as haircuts have to increase to protect the lender when volatility is high. To capture this link, we write the fraction of the funding gap covered by a loan \( \ell_t \) on the collateralized money market \( \alpha_t \) as:

\[
\alpha_t = \min \left\{ \frac{\ell_t}{\sigma^d d_t - m_t}, 1 \right\}.
\]

To borrow \( \ell_t \) on the collateralized money market, we impose a value-at-risk constraint. The annualized probability that the collateral value becomes lower than the value of loan \( \ell_t \) has to be at most \( p \).\(^{12}\) The quantity of collateral \( \chi_t \) required to borrow \( \ell_t \) in the interbank market has to satisfy:

\[
P \left[ \chi_t \exp \left( \int_{t}^{t+1} (\mu^*_u - (\sigma^*_u)^2/2 + \sigma^*_u (Z_{t+1} - Z_t)) du \right) \leq \ell_t \right] = p.
\]

\(^{12}\)The value-at-risk constraint is evaluated assuming that the drift \( \mu^*_t \) and volatility \( \sigma^*_t \) are constant. That is, bankers approximate

\[
P \left[ \chi_t \exp \left( \int_{t}^{t+1} (\mu^*_u - (\sigma^*_u)^2/2) du + \int_{t}^{t+1} \sigma^*_u dZ_u \right) \leq \ell_t \right] = p
\]

with equation (22). Also, for parsimony, we do not keep track of the distribution of collateral amongst banks.
Thus, if a fraction $\kappa^\chi$ of the securities held by the bank can be used as collateral, the quantity of available collateral is given by

$$\chi_t = \kappa^\chi q_t s_t.$$  \hfill (23)

Combining (22) and (23), the maximum amount that can be borrowed on the collateralized money market is given by:

$$\ell_t = \kappa^\ell q_t s_t,$$

where $\kappa^\ell = \kappa^\chi \exp \left( \Phi^{-1} (p) \sigma_t^s + \mu_t^s - (\sigma_t^s)^2 / 2 \right)$.

When solving numerically our model, we use the functional forms derived in (22) and (23).

### 4.3 Collateral Scarcity Spiral

Our model features a collateral spiral à la Brunnermeier and Pedersen (2009). As a series of adverse shocks hit the economy, the wealth of financial intermediaries decreases, and asset prices drop. This sharp decline in asset prices increases the endogenous volatility of the economy $\sigma_t^2$, which impacts haircuts in money markets. At some point, the economy enters the collateral scarcity regime (when $\alpha_t < 1$) and the deterioration in money market conditions results in more liquidity risk, a higher drop in asset prices, and a further drop in the wealth of bankers. This link between the wealth of financial intermediaries, endogenous volatility, and haircuts create a self-reinforcing downward spiral illustrated in

**Figure 5:** provides a schematic description of the feedback effect coming out of a negative realization of the Brownian shock when collateral is scarce.
Proposition 10.

**Proposition 10 (Amplification)** Without monetary policies, the endogenous volatility of the state variables $\eta_t$ and $\bar{\eta}_t$ are given by:

$$
\sigma^\eta_t \eta_t = \frac{(\nu_t - \eta_t)\sigma}{1 - \frac{q_t}{q} (\nu_t - \bar{\eta}_t) - \frac{\bar{\sigma}_t}{q} (\nu_t - \bar{\eta}_t)},
$$

$$
\sigma^\bar{\eta}_t \bar{\eta}_t = \frac{(\nu_t - \bar{\eta}_t)\sigma}{1 - \frac{q_t}{q} (\nu_t - \eta_t) - \frac{\bar{\sigma}_t}{q} (\nu_t - \bar{\eta}_t)},
$$

where

$$q_\eta = \frac{\partial q(\eta_t, \bar{\eta}_t)}{\partial \eta_t}, \quad q_\bar{\eta} = \frac{\partial q(\eta_t, \bar{\eta}_t)}{\partial \bar{\eta}_t}.$$

As in Brunnermeier and Sannikov (2014), an amplification spiral arises because of a feedback loop between lower wealth of financial intermediaries and higher endogenous volatility (see Figure 5). This can be seen from the denominator of this equation that corresponds to the sum of two geometric series. The size of this amplification factor depends on the derivatives of the securities’ price function with respect to the two state variables.

Figure 6 displays the solution of the model as a function of the total share of wealth in hands of regular and shadow banks $\eta_t + \bar{\eta}_t$ along the diagonal line $\eta_t = \bar{\eta}_t$ when $\alpha_t$ is endogenously fixed to 1 and when it evolves endogenously according to constraint (22). The drop in asset prices arises at a faster pace with the collateral spiral cycle. The mechanism is triggered when collateral become scarce—$\alpha_t$ drops below one—and generates an increase in endogenous volatility and a drop in asset prices.

### 4.4 Monetary Policy in a Dynamic Setting

In this subsection, we investigate, in the fully dynamic setting, how the different monetary policies may partially counteract the collateral spiral. To do so, we present in figure 7 the impulse response functions of shock leading to a destruction of 30% of the wealth of the banking sector with and without policy intervention. The blue line shows how the price of securities $q_t$, the endogenous volatility $\sigma^q_t$, and the collateral scarcity $\alpha_t$ evolve through time after the initial shock without any monetary policy reaction. The red line
Figure 6: The figure shows the amplification mechanism when $\alpha_t$ is fixed to 1 (blue line) and $\alpha_t$ is endogenous (red line). The three panels display the model solution for the price of securities $q_t$, the endogenous volatility $\sigma^2_t$ and the index of money market functioning $\alpha_t$ as a function of the total share of wealth in hands of regular and shadow banks $\eta_t + \bar{\eta}_t$ along the diagonal line $\eta_t = \bar{\eta}_t$.

shows the same variables when the central bank reacts to the shock by an increase in the supply of the reserves from $m = 0$ to $m = 0.5$ (liquidity injection policy), enough to satiate the traditional banks. Any further increase in money would, therefore, not change the equilibrium anymore as reserves are Wallace neutral from this point. The yellow line shows how the variables evolve if the central bank decides to complement its liquidity injection policy by an asset purchase policy by increasing its holding of securities from $\nu = 0$ to $\nu = 0.25$. The result derived in the static model, that asset purchase policies may have an impact on the economy when liquidity injections do not, holds in the fully dynamic setting.

5 Conclusion

In this article, we propose a path for introducing funding liquidity risk in a general equilibrium intermediary asset pricing model. With inspirations from the monetary policy implementation literature, we do so by assuming that leveraging by issuing short-term liabilities to hold long-term capital market assets generate liquidity risk. The framework allows us to study the benefits and limitations of three conceptually different types of monetary policy. Our analysis concludes that the most forceful policy mix implies to first use discount window and liquidity injection policies to alleviate funding stresses up to the point where traditional banks are fully satiated. If the shadow banking sector is large,
Figure 7: The figure displays the impulse response function for a 30% drop in the wealth of regular and shadow bankers. More precisely, starting from the stochastic steady-state, we plot the average impulse response functions for $q_t$, $\sigma^2_t$, and $\alpha_t$ after a shock to securities $dZ_t$ that destroys 30% of the stock of securities. The blue line corresponds to a no monetary policy benchmark. The red line corresponds to the shock accompanied by an increase of reserves from $m = 0$ to $m = 0.5$ (liquidity injection policy). The yellow line corresponds to the same rise in reserves accompanied by an increase in central bank asset purchases from $\nu = 0$ to $\nu = 0.25$ (liquidity injection policy and asset purchase policy).

This may not be sufficient to address all of the downward pressures in asset prices. In this case, the only tool available to go further is for the central bank to directly purchase long-term assets. This suggests that, even when these are costly, they can be beneficial for the economy in contexts in which money markets are impaired and the shadow banking sector is large. Overall, this article points out the importance of understanding how the development of a more international financial system leads central banks to extend their set of policy tools to address systemic liquidity crises.
References

Adrian, Tobias and Ashcraft, Adam B. Shadow banking regulation. Staff Reports 559, Federal Reserve Bank of New York, April 2012.


Appendices

A Omitted Derivations

Regular Banks

We first write the Hamilton-Jacobi-Bellman (HJB) equation of traditional bankers’ problem:

\[ 0 = \max_{w_t^s, w_t^d, w_t^m, c_t} f(c_t) + E_t(dV_t) \]

Apply Ito’s lemma, we have:

\[ E_t(dV_t) = V_t \xi_t + V_t \mu_t n_t + \frac{1}{2} \left[ V_{\xi\xi_t} (\xi_t^2) + V_{\mu \mu_t} n_t^2 \right. \]

\[ + 2V_{\xi \mu_t} \xi_t \mu_t n_t \left. \right] \]

Deriving our guess function and substituting in the former equations, we can simplify the HJB into:

\[ 0 = \max_{w_t^s, w_t^d, w_t^m, c_t} f(c_t) + (\xi_t n_t)^{1-\gamma} \left[ \mu_t \xi_t + r_t^b + w_t^s (\mu_t - r_t^b) + w_t^m (r_t^m - r_t^b) - w_t^d (r_t^d - r_t^b) - c_t + \mu_t \right. \]

\[ - \frac{\gamma}{2} \left( (\xi_t^2) + (w_t^s \xi_t^2)^2 + (\theta_t (w_t^d \sigma_t - w_t^m)) \right) \left. - 2 \frac{1-\gamma}{\gamma} \xi_t \left( w_t^s \xi_t + \theta_t (w_t^d \sigma_t - w_t^m) \right) \right] \]

Note that the maximum function bounding the liquidity risk to being non-negative does not appear in the previous equations. We treat this kink by solving for the optimality conditions first when the maximum function is not binding and then when it is binding by simply setting \( \theta_t (w_t^d \sigma - w_t^m) = 0 \). We apply the maximum principle, and combine the FOCs for the two regions in equations (8), (12), (10) and (13). The fact that \( V \) is non-differentiable at the kink does not prevent the existence of a (viscosity) solution to the optimisation problem.

Shadow Banks

The optimization problem of shadow banks is nested by the problem of regular banks assuming that \( w_t^m = 0 \) and \( \lambda_t = \lambda \) if fixed. Solving this problem yields the FOCs given in equations (9), (11) and (14).
Households  Similarly, households’ problem is nested when restricted to only hold risk-free deposits as a means of saving. The unique FOC of this problem is given by equation (15).

B  Proofs

B.1  Solving the Static Model

We guess and verify the static equilibrium by setting $\sigma^q = \sigma^\xi = \sigma^{\xi,h} = 0$ as well as $\mu^q = \mu^\xi = \mu^{\xi,h} = 0$. We start from plugging back each agent’s FOCs into its HJB equation.

For regular bankers:

$$0 = \frac{c - \rho}{1 - 1/\zeta} + r^d + w^s \gamma \theta^2 \sigma^d \left( \sigma^d w^d - w^m \right) + w^m \gamma \theta^2 (\sigma^d - 1) \left( \sigma^d w^d - w^m \right) - (w^s + w^m - 1 - w^d) \gamma \theta^2 \sigma^d \left( \sigma^d w^d - w^m \right) - c + \mu^\tau + \gamma w^s \sigma \left( w^s \sigma + \sigma^\tau \right) - 1/2 \gamma (w^s \sigma + \sigma^\tau)^2 - 1/2 \gamma \theta^2 \left( \sigma^d w^d - w^m \right)^2$$

After some algebra, we have:

$$0 = \frac{\bar{c} - \rho}{1 - 1/\zeta} + \frac{a}{q} + \Phi - \gamma \sigma (\bar{w}^s \sigma + \bar{\sigma}^\tau) + 1/2 \gamma \theta^2 \left( \sigma^d \bar{w}^d \right)^2 + 1/2 \gamma (\bar{w}^s \sigma)^2 - 1/2 \gamma (\bar{\sigma}^\tau)^2 - \bar{c} + \bar{\mu}^\tau$$

For shadow bankers:

$$0 = \frac{\bar{c} - \rho}{1 - 1/\zeta} + \frac{a}{q} + \Phi - \gamma \sigma (\bar{w}^s \sigma + \bar{\sigma}^\tau) + 1/2 \gamma (\bar{w}^s \sigma)^2 - 1/2 \gamma (\bar{\sigma}^\tau)^2 + 1/2 \gamma (\bar{\sigma}^d \bar{\theta})^2 - \bar{c} + \bar{\mu}^\tau$$

For households:

$$0 = \frac{c^h - \rho}{1 - 1/\zeta} + r^d - c^h$$

We solve for endogenous equilibrium portfolio choices. First, we rewrite equation (16) as:

$$r^d = \frac{a}{q} + \Phi - \gamma \sigma (w^s \sigma + \sigma^\tau) - \gamma \theta^2 \sigma^d \left( \sigma^d w^d - w^m \right)$$
and similarly for shadow banks:

\[ r^d = \frac{\alpha}{q} + \Phi - \gamma \sigma (\bar{w}^s \sigma + \sigma^T) - \gamma \bar{\theta}^d \bar{w}^d \left( \sigma^d \right)^2 \]

We then equalize the two equations:

\[ \sigma (w^s \sigma + \sigma^T) + \bar{\theta}^d \sigma^d \left( \sigma^d w^d - w^m \right) = \sigma (\bar{w}^s \sigma + \bar{\sigma^T}) + \bar{w}^d \left( \sigma^d \bar{\theta}^d \right)^2 \]

After some algebra, we have:

\[ \bar{\pi}^s (\sigma^2 + (\sigma^d \bar{\theta})^2) = w^s (\sigma^2 + (\sigma^d \theta)^2) + w^s (\sigma^d \theta)^2 - w^m \theta^2 \sigma^d + (\sigma^d)^2 (\bar{\theta}^2 - \theta^2) + \sigma (\sigma^T - \bar{\sigma^T}) \]

Note that since we have:

\[ \bar{\pi}^s = \frac{w^s \sigma^2 + (\sigma^d \theta)^2 \sigma^2 + (\sigma^d \theta)^2}{\sigma^2 + (\sigma^d \theta)^2} + \frac{w^s (\sigma^d \theta)^2 - w^m \theta^2 \sigma^d + (\sigma^d)^2 (\bar{\theta}^2 - \theta^2) + \sigma (\sigma^T - \bar{\sigma^T})}{\sigma^2 + (\sigma^d \theta)^2}, \]

any transfer rule such that \( \sigma^T = \bar{\sigma^T} \) renders asset purchases neutral in the absence of liquidity risk.

For parsimony, let’s define \( \bar{\kappa}^w = 1/\kappa^w \), \( \bar{\kappa} = \kappa / \bar{\kappa}^w \) such that:

\[ \bar{\pi}^s = \frac{w^s \sigma^2 + (\sigma^d \theta)^2}{\sigma^2 + (\sigma^d \theta)^2} + \frac{w^s (\sigma^d \theta)^2 - w^m \theta^2 \sigma^d + (\sigma^d)^2 (\bar{\theta}^2 - \theta^2)}{\sigma^2 + (\sigma^d \theta)^2} \]

\[ = \kappa^w w^s + \kappa \]

From the securities market clearing condition (1), we have:

\[ w^\eta + \bar{w}^\eta + w^s \eta = 1 \]
\[ w^s \eta + \kappa^w w^s \bar{\eta} + \kappa \bar{\eta} + w^s \eta = 1 \]
\[ w^s = \frac{1 - \kappa \bar{\eta} - w^s \eta}{\eta + \kappa^w \bar{\eta}} \]
$$w^s = \frac{1 - w^s \eta}{\eta} - \frac{1 - \kappa \eta - w^s \eta \eta}{\eta + \kappa \eta \eta}$$

$$= \eta + \kappa \eta (\eta + \kappa \eta) w^s \eta - \eta + \kappa \eta \eta + w^s \eta \eta$$

$$= \frac{1 - w^s \eta + \kappa \eta}{\kappa \eta + \eta}$$

**Consumption Market Clearing**

$$c \eta + \tilde{c} \eta + \left( \frac{\rho}{1/\zeta} - \gamma \sigma^d \frac{1 - 1/\zeta}{1/\zeta} \right) (1 - \eta - \bar{\eta}) = \frac{a}{q}$$

After some algebra:

$$\left( c \eta + \tilde{c} \eta \right) \frac{1/\zeta}{1 - 1/\zeta} = \frac{a}{q} \left( \frac{1/\zeta + (1 - \eta - \bar{\eta})(1 - 1/\zeta)}{1 - 1/\zeta} \right)$$

$$- \left( \frac{\rho}{1 - 1/\zeta} - \left( \Phi - \gamma \sigma (w^s \sigma + \sigma^r) - \gamma \theta^2 \sigma^d \left( \sigma^d \omega^d - \omega^m \right) \right) \right) (1 - \eta - \bar{\eta})$$

**HJBs** We can now plug in all derived variables into the respective HJB equations and take the sum of the three of them.

$$0 = \frac{c - \rho}{1 - 1/\zeta} + \frac{a}{q} \Phi - \gamma \sigma (w^s \sigma + \sigma^r) + 1/2 \gamma \theta^2 \left( \sigma^d \omega^d - \omega^m \right)^2 + 1/2 \gamma (w^s \sigma)^2 - 1/2 \gamma (\sigma^r)^2$$

$$- c + \mu^r$$

$$0 = \frac{a}{q} \frac{1}{1 - 1/\zeta} - \frac{\rho}{1 - 1/\zeta}$$

$$+ \Phi - \gamma \sigma (w^s \sigma + \sigma^r) - \gamma \theta^2 \sigma^d \left( \sigma^d \omega^d - \omega^m \right) (1 - \eta - \bar{\eta}) - \gamma \theta^2 \sigma^d \left( \sigma^d \omega^d - \omega^m \right) \bar{\eta} + \gamma \bar{\omega}^d \left( \sigma^d \bar{\eta}^2 \right) \bar{\eta}$$

$$+ \left( 1/2 \gamma \theta^2 \left( \sigma^d \omega^d - \omega^m \right)^2 + 1/2 \gamma (w^s \sigma)^2 - 1/2 \gamma (\sigma^r)^2 + \mu^r \right) \eta$$

$$+ \left( 1/2 \gamma \theta^2 \left( \sigma^d \omega^d \right)^2 + 1/2 \gamma (w^s \sigma)^2 - 1/2 \gamma (\sigma^r)^2 + \mu^r \right) \bar{\eta}$$
since

\[
\sigma(w^s \sigma + \sigma^\tau) + \theta^2 \sigma^d \left( \sigma^d w^d - w^m \right) = \sigma(\overline{w}^s \sigma + \overline{\sigma}^\tau) + \overline{\sigma}^d \left( \sigma^d \overline{\theta} \right)^2
\]

\[
0 = \frac{a}{\eta} \frac{1}{1 - 1/\zeta} - \frac{\rho}{1 - 1/\zeta}
\]

\[
+ \Phi - \gamma \sigma (w^s \sigma + \sigma^\tau) - \gamma \theta^2 \sigma^d \left( \sigma^d w^d - w^m \right) (1 - \eta) + \gamma \overline{\sigma}^d \left( \sigma^d \overline{\theta} \right)^2 \eta
\]

\[
+ \left( \frac{1}{2} \gamma \theta^2 \left( \sigma^d w^d - w^m \right)^2 + \frac{1}{2} \gamma (w^s \sigma)^2 + \frac{1}{2} \gamma (\sigma^\tau)^2 + \mu^\tau \right) \eta
\]

\[
+ \left( \frac{1}{2} \gamma \overline{\theta}^2 \left( \sigma^d \overline{\sigma}^d \right)^2 + \frac{1}{2} \gamma (\overline{w}^s \sigma)^2 + \frac{1}{2} \gamma (\overline{\sigma}^\tau)^2 + \overline{\mu}^\tau \right) \eta
\]
Note that if \( \theta = \bar{\theta} \) and no policies; then \( w^* = \bar{w}^* = 1/(\eta + \bar{\eta}) \)

\[
0 = \frac{a}{q} \left( \frac{1}{1 - \frac{1}{\zeta}} - \frac{\rho}{1 - \frac{1}{\zeta}} \right) + \Phi - \frac{\gamma}{\sigma^2} (w^* \sigma) - \gamma (\theta \sigma^d)^2 w^d (1 - \eta) + \gamma \bar{w}^d \left( \sigma^d \bar{\theta} \right)^2 \bar{\eta} + 1/2 \gamma \left( \theta^2 \left( \sigma^d \theta^d \right)^2 + (w^* \sigma)^2 \right) (\eta + \bar{\eta})
\]

\[
= \frac{a}{q} \left( \frac{1}{1 - \frac{1}{\zeta}} - \frac{\rho}{1 - \frac{1}{\zeta}} \right) + \Phi - \frac{1}{2} \gamma w^* \sigma^2 - \gamma (\theta \sigma^d)^2 w^d (1 - \eta - \bar{\eta}) + 1/2 \gamma \left( \sigma^d \theta^d \right)^2 (\eta + \bar{\eta})
\]

\[
= \frac{a}{q} \left( \frac{1}{1 - \frac{1}{\zeta}} - \frac{\rho}{1 - \frac{1}{\zeta}} \right) + \Phi - \frac{1}{2} \gamma w^* \sigma^2 - \gamma (\theta \sigma^d)^2 (w^* - 1)(1 - \eta - \bar{\eta}) + 1/2 \gamma \left( \sigma^d \theta^d \right)^2 ((w^*)^2 + 1 - 2w^*)(\eta + \bar{\eta})
\]

\[
= \frac{a}{q} \left( \frac{1}{1 - \frac{1}{\zeta}} - \frac{\rho}{1 - \frac{1}{\zeta}} \right) + \Phi - \frac{1}{2} \gamma w^* \sigma^2 - \gamma (\theta \sigma^d)^2 w^* (1 - \eta - \bar{\eta}) + \gamma (\theta \sigma^d)^2 (1 - \eta - \bar{\eta}) + 1/2 \gamma \left( \sigma^d \theta^d \right)^2 (\eta + \bar{\eta}) - \gamma \left( \sigma^d \theta^d \right)^2
\]

\[
= \frac{a}{q} \left( \frac{1}{1 - \frac{1}{\zeta}} - \frac{\rho}{1 - \frac{1}{\zeta}} \right) + \Phi - \frac{1}{2} \gamma w^* \sigma^2 - \gamma (\theta \sigma^d)^2 w^* (1 - \eta - \bar{\eta}) + \gamma (\theta \sigma^d)^2 + 1/2 \gamma \left( \sigma^d \theta^d \right)^2 (\eta + \bar{\eta}) - \gamma \left( \sigma^d \theta^d \right)^2
\]

\[
= \frac{a}{q} \left( \frac{1}{1 - \frac{1}{\zeta}} - \frac{\rho}{1 - \frac{1}{\zeta}} \right) + \Phi - \frac{1}{2} \gamma w^* \sigma^2 - \gamma (\theta \sigma^d)^2 w^* + \gamma (\theta \sigma^d)^2 - 1/2 \gamma \left( \sigma^d \theta^d \right)^2 (\eta + \bar{\eta})
\]

\[
= \frac{a}{q} \left( \frac{1}{1 - \frac{1}{\zeta}} - \frac{\rho}{1 - \frac{1}{\zeta}} \right) + \Phi - \frac{1}{2} \gamma w^* (\sigma^2) - 1/2 \gamma (\sigma^d \theta^d)^2 \left( \eta + \bar{\eta} + \frac{1}{\eta + \bar{\eta}} \right) + \gamma (\sigma^d \theta^d)^2
\]
which is consistent with previous results. We then have:

\[
0 = \frac{a}{q \frac{1}{1-1/\zeta}} - \frac{\rho}{1-1/\zeta} \\
+ \Phi - \gamma \sigma (w^s \sigma + \sigma^\tau) - \gamma \theta^2 \sigma^d \left( \sigma^d w^d - w^m \right) (1 - \eta) + \gamma w^d \left( \sigma^d \theta \right)^2 \bar{\eta} \\
+ \left( \frac{1}{2} \gamma \theta^2 \left( \sigma^d w^d - w^m \right)^2 + \frac{1}{2} \gamma (w^s \sigma)^2 - \frac{1}{2} \gamma (\sigma^\tau)^2 + \mu^\tau \right) \eta \\
+ \left( \frac{1}{2} \gamma \bar{\theta}^2 \left( \sigma^d \bar{w}^d \right)^2 + \frac{1}{2} \gamma (\bar{w}^s \sigma)^2 - \frac{1}{2} \gamma (\bar{\sigma}^\tau)^2 + \bar{\mu}^\tau \right) \bar{\eta}
\]

\[
0 = \frac{a}{q \frac{1}{1-1/\zeta}} - \frac{\rho}{1-1/\zeta} \\
+ \Phi - \frac{1}{2} \gamma w^s \sigma^2 - \frac{1}{2} \gamma \theta^2 \sigma^d \left( \sigma^d w^d - w^m \right) (1 - \eta) + \frac{1}{2} \gamma w^d \left( \sigma^d \theta \right)^2 \bar{\eta} \\
+ \frac{1}{2} \gamma \left( \theta^2 \left( w^m \right)^2 \eta - \theta^2 \sigma^d w^d w^m \eta - w^s \sigma^2 \bar{w}^s \eta \right) \\
+ \mu^\tau \eta + \bar{\mu}^\tau \bar{\eta} - \frac{1}{2} \gamma (\sigma^\tau)^2 (\eta + \bar{\eta}) - \gamma \sigma \sigma^\tau
\]

\[
\sigma^\tau = \frac{\sigma w^s \eta}{\eta + \bar{\eta}}
\]

\[
\mu^\tau \eta = \left( r^b - r^m \right) (w^m - w^s \eta) + \frac{\eta}{\eta + \bar{\eta}} \left( \mu^s - r^d \right) w^s \eta + \left( r^d - r^m \right) w^s \eta
\]

\[
\bar{\mu}^\tau \bar{\eta} = \frac{\bar{\eta}}{\eta + \bar{\eta}} \left( \mu^s - r^d \right) w^s \eta
\]

\[
r^b - r^m = \gamma \theta^2 (\sigma^d w^d - w^m)
\]

\[
\mu^s - r^b = \gamma \sigma (w^s \sigma + \sigma^\tau)
\]
\[ \mu^r \eta + \mu^r \bar{\eta} = (r^b - r^m) w^m \eta + (\mu^s - r^b) w^s \eta \\
= \gamma \theta^2 (\sigma^d w^d - w^m) w^m \eta + \gamma \sigma (w^s \sigma + \sigma^r) w^s \eta \]

Asset Purchase Policy \( \sigma^d = 1, \theta = \bar{\theta}, w^s = \bar{w}^m \)

Thus \( w^s = \bar{w}^s = \frac{1-w^s \eta}{\eta + \bar{\eta}} \)

\[
0 = \frac{a}{q} \frac{1}{1 - 1/\zeta} - \frac{\rho}{1 - 1/\zeta} \\
+ \Phi - 1/2 \gamma \frac{\sigma^2}{\eta + \bar{\eta}} - 1/2 \gamma \theta^2 \left( \frac{1}{\eta + \bar{\eta}} - 1 \right) (1 - \eta - \bar{\eta}) \\
+ 1/2 \gamma \theta^2 w^s \eta \left( \frac{1}{\eta + \bar{\eta}} - \frac{1}{\eta + \bar{\eta} + 2} \right)
\]

After some algebra, we have:

\[
0 = \frac{a}{q} \frac{1}{1 - 1/\zeta} - \frac{\rho}{1 - 1/\zeta} \\
+ \Phi - 1/2 \gamma \frac{\sigma^2}{\eta + \bar{\eta}} - 1/2 \gamma \theta^2 \left( \frac{1}{\eta + \bar{\eta}} - 1 \right) (1 - \eta - \bar{\eta}) \\
+ 1/2 \gamma \theta^2 w^s \eta \left( \frac{1}{\eta + \bar{\eta}} - \frac{1}{\eta + \bar{\eta} + 2} \right)
\]

\[
q = \frac{a}{\rho - (1 - 1/\zeta) \left( \Phi - 1/2 \gamma \frac{\sigma^2}{\eta + \bar{\eta}} - 1/2 \gamma \theta^2 \left( \frac{1}{\eta + \bar{\eta}} \right)^2 \right)}
\]
Liquidity Injection Policy $\sigma^d = 1$, $\theta = \bar{\theta}$, $w^s = 0$

Thus,

$$w^s = \frac{1}{\eta + \bar{\eta}} + \frac{w^m \theta^2}{\sigma^2 + \theta^2} \frac{\eta}{\eta + \bar{\eta}}$$

$$\bar{w}^s = \frac{1}{\eta + \bar{\eta}} - \frac{w^m \theta^2}{\sigma^2 + \theta^2} \frac{\eta}{\eta + \bar{\eta}}$$

$$0 = \frac{a}{q} \frac{1}{1 - 1/\zeta} - \frac{\rho}{1 - 1/\zeta}$$

$$+ \Phi - 1/2\gamma w^s \sigma^2 - 1/2\gamma \theta^2 \sigma^d \left( \sigma^d w^d - w^m \right) (1 - \eta) + 1/2\gamma \bar{w}^d \left( \sigma^d \bar{\theta} \right)^2 \bar{\eta}$$

$$+ 1/2\gamma \left( \theta^2 (w^m)^2 \eta - \theta^2 \sigma^d w^d w^m \eta - w^s \sigma^2 w^s \eta \right)$$

$$+ \gamma \theta^2 (\sigma^d w^d - w^m)w^m \eta + \gamma \sigma (w^s \sigma + \sigma^2) w^s \eta$$

$$- 1/2\gamma \sigma^2 \left( \frac{w^s \eta}{\eta + \bar{\eta}} \right)^2 (\eta + \bar{\eta}) - \gamma \sigma^2 \left( \frac{w^s \eta}{\eta + \bar{\eta}} \right)$$

After some algebra this simplifies into:

$$0 = \frac{a}{q} \frac{1}{1 - 1/\zeta} - \frac{\rho}{1 - 1/\zeta}$$

$$+ \Phi - 1/2\gamma \frac{\sigma^2}{\eta + \bar{\eta}}$$

$$- 1/2\gamma \theta^2 \left( \frac{1}{\eta + \bar{\eta}} - 1 \right) (1 - \eta - \bar{\eta} - 2w^m \eta)$$

$$- 1/2\gamma (w^m \theta)^2 \eta \left( \frac{\eta \sigma^2 + \eta \theta^2 + \bar{\eta} \sigma^2}{(\sigma^2 + \theta^2)(\eta + \bar{\eta})} \right)$$

$$- 1/2\gamma \theta^2 (w^m)^2 \eta$$

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\[ 0 = \frac{a}{q} \frac{1}{1 - 1/\zeta} - \frac{\rho}{1 - 1/\zeta} + \Phi - 1/2 \gamma \frac{\sigma^2}{\eta + \bar{\eta}} \left( 1 + \frac{(w^m \eta \theta)^2 \bar{\eta}}{\sigma^2 + \theta^2 \eta} \right) - 1/2 \gamma \theta^2 \left( \frac{1 + (w^m \eta)^2 - 2w^m \eta}{\eta + \bar{\eta}} + \eta + \bar{\eta} - 2(1 - w^m \eta) \right) \]

\[ 0 = \frac{a}{q} \frac{1}{1 - 1/\zeta} - \frac{\rho}{1 - 1/\zeta} + \Phi - 1/2 \gamma \frac{\sigma^2}{\eta + \bar{\eta}} \left( 1 + \frac{m^2 \theta^2 \bar{\eta}}{\sigma^2 + \theta^2 \eta} \right) - 1/2 \gamma \theta^2 \left( \frac{1 + m^2 - 2m}{\eta + \bar{\eta}} + \eta + \bar{\eta} - 2(1 - m) \right) \]

where \( w^m n = M = mqS \)

\[ q = \frac{a}{\rho - (1 - 1/\zeta) \left( \Phi - 1/2 \gamma \frac{\sigma^2}{\eta + \bar{\eta}} \left( 1 + \frac{m^2 \theta^2 \eta}{\sigma^2 + \theta^2 \eta} \right) - 1/2 \gamma \theta^2 \left( \frac{1 + m^2 - 2m}{\eta + \bar{\eta}} + \eta + \bar{\eta} - 2(1 - m) \right) \right)} \]

Let’s not forget that these equations are valid only if \( m \leq w^d = (w^s - 1)\eta \) in the case of reserves and \( 0 \leq (w^s - 1)\eta \) in the case of QE. That is,

\[ w^m \leq \frac{1}{\eta + \bar{\eta}} + \frac{w^m \eta \theta^2 \bar{\eta}}{\sigma^2 + \theta^2 \eta + \bar{\eta}} - 1 \]

Equality arises if

\[ m = \frac{\eta}{\eta + \bar{\eta}} + \frac{m \theta^2 \eta}{\sigma^2 + \theta^2 \eta + \bar{\eta}} - \eta \]

\[ m \left( 1 - \frac{\theta^2}{\sigma^2 + \theta^2 \eta + \bar{\eta}} \right) = \frac{\eta}{\eta + \bar{\eta}} - \eta \]

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\[
m \left( \frac{\sigma^2 \eta + \theta^2 \eta + \sigma^2 \bar{\eta} + \theta^2 \bar{\eta} - \theta^2 \bar{\eta}}{(\sigma^2 + \theta^2)(\eta + \bar{\eta})} \right) = \frac{\eta}{\eta + \bar{\eta}} - \eta
\]

\[
m \left( \frac{\sigma^2 \eta + \theta^2 \eta + \sigma^2 \bar{\eta}}{(\sigma^2 + \theta^2)(\eta + \bar{\eta})} \right) = \frac{\eta - \eta^2 - \eta \bar{\eta}}{\eta + \bar{\eta}}
\]

\[
m = \frac{(\eta - \eta^2 - \eta \bar{\eta})(\sigma^2 + \theta^2)}{\sigma^2 \eta + \theta^2 \eta + \sigma^2 \bar{\eta}}
\]

which is exactly \(1/2 \times m^*\) where \(q(m^*) = q(0)\); which makes sense as the risk with no reserves is \(w^d \theta\) and the risk with \(m^* = 2w^d \eta\) is \((w^d - 2w^d \eta) \theta = -w^d \theta\). Also,

\[
m = \frac{(\eta - \eta^2 - \eta \bar{\eta})(\sigma^2 + \theta^2)}{\sigma^2 \eta + \theta^2 \eta + \sigma^2 \bar{\eta}}
\]

\[
= (1 - \eta - \bar{\eta}) \frac{\sigma^2 + \theta^2}{\sigma^2 + \theta^2 + \sigma^2 \bar{\eta} / \eta} < 1 - \eta - \bar{\eta}
\]
C Micro-Foundations for Funding Liquidity Risk

In this appendix, we provide micro-foundation for our formulation of liquidity risk depending on the functioning of money markets and central bank reserves held for precautionary motives. The environment presented here is a simplified version of Afonso and Lagos (2015) and Bianchi and Bigio (2014) with the following assumptions. First, the idiosyncratic deposit shock is assumed to follow a binomial distribution with even probabilities. Second, all banks are assumed to be price taker in money markets and are required to trade at the current risk free-rate provided there is enough collateral to do so. Third, securities can be used for settlement in the last round of the interim period but at an exogenously given discount with respect to the fundamental value of securities (its value in the active trading stage).

With the given restrictions, the model can be decomposed between two stages: the active stage and the passive stage. In the active stage, managers take their portfolio decision in order to maximize the life-time utility of their shareholders knowing that potential developments during the passive stage may arise. This is the period we consider in our continuous time specification. In the passive stage, the money market desk of the bank adjusts the balance sheet following given rules in order to adjust to balance sheet identity by the end of the day. Managers, therefore, have to take portfolio decisions taking into account the distribution of funding shocks and their impact on profit flows given conditions in money markets.

The sequence of balance sheet adjustments is described in figure 8.

1. At time $t$, the two banks considered are in the active stage and decide optimally of their balance sheet (quantity of securities $S$, reserves holdings $R$ and deposits $D$, given an initial level of equity $E$).

2. The passive stage starts and the funding shock reshuffles deposits from the deficit bank to the surplus bank; creating, respectively, a liquidity deficit and a liquidity surplus.

3. Reserves are transferred from the deficit bank to the surplus bank as the prime means of settlement between banks.

4. Money market opens and the deficit bank acquires as much of its funding gap as made
possible by the amount of available collateral. In this simple example, it receives the money market funding (MM) from the surplus bank to which the deposits have been reshuffled. It does not necessarily have to be case and the deficit bank could receive fundings from any bank in the economy in a centralized Walrasian market at the risk-free rate.

5. If the money market desk has not been able to fill its liquidity need, it has to resort to asking for a central bank loan at the discount window (DW).

6. If for some reasons, discount window loans are not available (i.e. being a shadow bank and not having access to it, or being a bank and not having enough eligible collateral), the bank has no other choice but to sell some of its securities at a discount with respect to its fundamental value and to net the corresponding loss in the next active period.

C.1 Poisson Shock

Liquidity Risk without Reserves

Upon the arrival of a funding shock on deposits, asset allocations cannot be changed and funding gaps need to be covered in the interbank money market. A quantity \( \sigma^d d_t \) of deposits are reshuffled from a deficit bank (receiving a negative shock) to a surplus bank (receiving a positive shock).

The deficit bank receives an intra-day credit from the clearing house which allows to temporarily cover its deficit with respect to the surplus bank. Then banks meet in the interbank money market and each deficit bank is matched to a surplus bank. The deficit banks use their securities as collateral to borrow as much as possible from the surplus banks at the risk free rate \( r^b_t \) for a period of time \( \Delta_d \) corresponding to a day.\(^{13}\) To cover the remaining part of the funding gap, banks need to fire-sale assets at the cost \( \lambda > 0 \).

For parsimony of the model, these transfers are instantaneous such that we do not need to keep track of the distribution of banks’ loans of maturity \( \Delta_d \). Thus, the transfer of wealth

\(^{13}\)We assume that bankers evaluate the rate \( r^b_t - r^d_t \) as constant over the short period of time \( \Delta_d \), that is:

\[
\int_t^{t+\Delta_d} r^b_u du \approx r^b_t \Delta_d.
\]

Because \( \Delta_d \) is corresponds to a short period of time (one day) and these idiosyncratic transfers aggregate to zero, this approximation has virtually no impact on the results.

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<table>
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<tr>
<th>Initial Position</th>
<th>Deposit Shock</th>
<th>Active Stage</th>
<th>Passive Stage</th>
<th>Active Stage</th>
<th>Active Stage</th>
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<td>Deficit Bank</td>
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<td>Surplus Bank</td>
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<td>Surplus Bank</td>
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<td>Surplus Bank</td>
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</tr>
</tbody>
</table>

**Figure 8:** Sequence of Balance Sheet Adjustments
from a deficit bank to surplus bank follows:

\[
\alpha_t(r^b_t - r^d_t) \Delta d + (1 - \alpha_t) \lambda \sigma^d d_t,
\]

where \( N_t \) is a Poisson shock with arrival intensity \( \phi \). The fraction of the funding gap covered by a loan \( \ell_t \) on the collateralized money market \( \alpha_t \) is given by

\[
\alpha_t = \min \left\{ \frac{\ell_t}{\sigma^d d_t}, 1 \right\}.
\]

Note that if \( \lambda = 0 \) and \( r^b_t = r^d_t \), the deposit shock does not impact banks’ net worth.

To borrow \( \ell_t \) on the collateralized money market, we impose a value-at-risk constraint. The annualized probability that the collateral value becomes lower than the value of loan \( \ell_t \) has to be at most \( p \).

The quantity of collateral \( \chi_t \) required to borrow \( \ell_t \) in the interbank market has to satisfy:

\[
P_t [\chi_t \exp (\sigma^d_t (Z_{t+1} - Z_t)) \leq \ell_t] = p.
\]

Thus, if a fraction \( \kappa^\chi \) of the securities held by the bank can be used as collateral, the quantity of available collateral is given by

\[
\chi_t = \kappa^\chi q_t s_t.
\]

Combining (27) and (28), the maximum amount that can be borrowed on the collateralized money market is given by:

\[
\ell_t = \kappa^\ell q_t s_t, \quad \text{where } \kappa^\ell_t = \kappa^\chi \exp \left( \Phi^{-1}(p) \sigma^d_t \right).
\]

**Liquidity Risk with Reserves** Negative deposit shocks can be absorbed by selling highly liquid central bank reserves. Assuming \( m_t < \sigma^d d_t \), the impact of a deposit shock

---

\(^{14}\)Similarly to footnote 13, the value-at-risk constraint is evaluated assuming that the volatility rate \( \sigma^t \) is constant. That is, bankers approximate

\[
P \left[ \chi_t \exp \left( \int_t^{t+1} \sigma_u^d dZ_u \right) \leq \ell_t \right] = p
\]

with equation (27). Also, for parsimony, we do not keep track of the distribution of collateral amongst banks.
becomes:

\[ m_t(r_t^m - r_t^d)\Delta_d + (\alpha_t(r_t^b - r_t^d)\Delta_d + (1 - \alpha_t)\lambda)(\sigma^d d_t - m_t) \]

where

\[ \alpha_t = \min\left\{ \frac{\ell_t}{\sigma^d d_t - m_t}, 1 \right\}. \]

C.2 Brownian Motion

Liquidity Risk  
Upon the arrival of a funding shock on deposits, asset allocations cannot be changed and funding gaps need to be covered in the interbank money market. A quantity \( \sigma^d d_t \) of deposits are reshuffled from a deficit bank (receiving a negative shock) to a surplus bank (receiving a positive shock).

The deficit bank receives an intra-day credit from the clearing house which allows to temporarily cover its deficit with respect to the surplus bank. Then banks meet in the interbank money market and each deficit bank is matched to a surplus bank. The deficit banks use their loans as collateral to borrow as much as possible from the surplus banks at the risk free rate \( r^b_t \) for a fixed period of time \( \Delta_d \) corresponding to a day.\(^{15}\) To cover the remaining part of the funding gap, banks need to fire-sale assets at the cost \( \lambda > 0 \).

Importantly, negative deposit shocks can be absorbed by selling highly liquid central bank reserves \( m_t \). Thus, if the bank is holding central bank reserves, the quantity to borrow in the interbank money market is reduced to \( \sigma^d d_t - m_t \). This idiosyncratic liquidity shock is represented using the Brownian motion \( d\tilde{Z}_t \).\(^{16}\) Thus, assuming \( m_t < \sigma^d d_t \), the transfer of

\(^{15}\)Implicitly, we assume that the rate \( r^b_t \) is constant over that short period of time \( \Delta_d \), that is:

\[ \int_t^{t+\Delta_d} r^b_u du \approx r^b_t \Delta_d. \]

Because \( \Delta_d \) is corresponds to a short period of time (one day) and these idiosyncratic transfers aggregate to zero, this approximation has virtually no impact on the results.

\(^{16}\)It is possible to represent this shock using either a Brownian motion or a Poisson shock. Both yield similar results; the Brownian motion yields simpler analytical results while the Poisson shock is more intuitive. In the benefit of exposure, we choose the Brownian motion. We construct the Brownian motion as the limit of a sum of \( n \) shocks of size \( \pm \theta_t \sqrt{\Delta t} \) with a fixed interval of time \( \Delta t \) as \( \Delta t \) tend to zero. This implies that, upon the arrival of a deposit shock of size \( \sigma^d d_t \sqrt{\Delta t} \), reserves can be reshuffled at the rate \( m_t \sqrt{\Delta t} \) and that collateral can be used at the rate \( \chi_t \sqrt{\Delta t} \). Without this assumption, as the size of the
wealth from a deficit bank to surplus bank follows:

$$\left[ m_t(r_t^m - r_t^d)\Delta_d + \left( \alpha_t(r_t^b - r_t^d)\Delta_d + (1 - \alpha_t)\lambda \right)(\sigma^d d_t - m_t) \right] d\tilde{Z}_t.$$  

These transfers of wealth are instantaneous instead of lasting from $t$ to $t + \Delta_d$ such that we do not have to keep track of the distribution of idiosyncratic shocks. The fraction of the funding gap covered by a loan $\ell_t$ on the collateralized money market $\alpha_t$ is given by

$$\alpha_t = \min \left\{ \frac{\ell_t}{\sigma^d d_t - m_t}, 1 \right\}.$$  

Note that if $\lambda = 0$ and $r_t^m = r_t^b = r_t^d$, the deposit shock does not impact banks’ net worth.

To borrow $\ell_t$ on the collateralized money market, we impose a value-at-risk constraint. The annualized probability that the collateral value becomes lower than the value of loan $\ell_t$ has to be at most $p$.\(^{17}\) The quantity of collateral $\chi_t$ required to borrow $\ell_t$ in the interbank market has to satisfy:

$$P \left[ \chi_t \exp \left( \mu_s - (\sigma_s^2/2 + \sigma_s^s (Z_{t+1} - Z_t)) \right) \leq \ell_t \right] = p. \quad (27)$$

Thus, if a fraction $\kappa^{\chi}$ of the securities held by the bank can be used as collateral, the quantity of available collateral is given by

$$\chi_t = \kappa^{\chi} q_t s_t. \quad (28)$$

Combining (27) and (28), the maximum amount that can be borrowed on the collateralized money market is given by:

$$\ell_t = \kappa^\ell q_t s_t, \quad \text{where} \quad \kappa^\ell = \kappa^{\chi} \exp \left( \Phi^{-1}(p) \sigma_s^s + \mu_s^s - (\sigma_s^s)^2/2 \right).$$

\(^{17}\)Similarly to footnote 13, the value-at-risk constraint is evaluated assuming that the drift $\mu_s^s$ and volatility $\sigma_s^s$ are constant. That is, bankers approximate

$$P \left[ \chi_t \exp \left( \int_{t}^{t+1} (\mu_u^s - (\sigma_u^s)^2/2)du + \int_{t}^{t+1} \sigma_u^s dZ_u \right) \leq \ell_t \right] = p$$

with equation (27). Also, for parsimony, we do not keep track of the distribution of collateral amongst banks.